Deep Dive into CDCL Pseudo-Boolean Solvers

focusing on the implementation in *Sat4j*

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Context

The CDCL Revolution

In the early 2000s, a revolution in the architecture of SAT solvers happened, with the wide adoption of the CDCL approach (Silva and Sakallah, 1996) and the use of efficient heuristics and data structures (Moskewicz et al., 2001; Eén and Sörensson, 2004)

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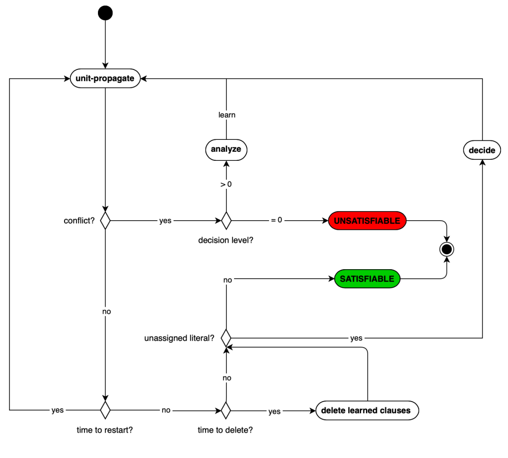
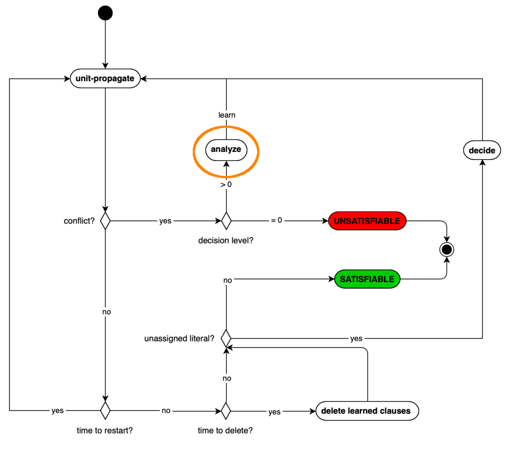
Modern SAT solvers can now deal with problems containing millions of variables and clauses



Overview of the CDCL Algorithm

Extending the CDCL Architecture



Use of the proof system in the CDCL Algorithm

Some CDCL Invariants

Important invariants of the CDCL algorithm in SAT solvers are:

- Constraints propagate only once
- Constraints have a single assertion level
- Combination of a reason and conflict leads to a conflict
- Syntactical assertion detection

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We will break these invariants with PB solvers!

Sat4j (Le Berre and Parrain, 2010)

- Open Source SAT solver library in Java developed since 2004
- Support for pseudo-Boolean solving and MAXSAT
- Native PB constraints support
- Various proof systems support with PB constraints
- Available from `http://sat4j.org/`

Pseudo-Boolean Constraints

Why using PB Constraints?

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While modern SAT solvers perform poorly on such instances for $n > 20$, PB solvers based on cutting-planes may solve them in linear time

Pseudo-Boolean (PB) Constraints

We consider conjunctions of linear equations or inequations over Boolean variables of the form:

$$\sum_{i=1}^{n} \alpha_i \ell_i \vartriangle \delta$$

in which

- the coefficients α_i are integers
- ℓ_i are literals, i.e., a variable v or its negation $\bar{v} = 1 - v$
- $\vartriangle$ is a relational operator among $\{<, \leq, =, \geq, >\}$
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For example:

$$-3a + 4b - 7c + d \leq -5$$

Normalized PB Constraints

Without loss of generality, we consider conjunctions of normalized PB constraints of the form:

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For example:

$$-3a + 4b - 7c + d \leq -5 \equiv 3a + 4\bar{b} + 7c + \bar{d} \geq 10$$

Fun Facts

The constraint $3a + 4\bar{b} + 7c + \bar{d} \geq 10$ propagates c to true

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The constraint above can be rewritten as $c \wedge 3a + 4\bar{b} + \bar{d} \geq 3$
but also as $c \wedge (a \vee \bar{b})$

A PB Encoding for Pigeonhole-Principle Formulae

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Let us see how to prove the unsatifiability of this formula

Example of a Human Proof ($n = 3$)

$$(1)\ p_{11} + p_{12} \geq 1$$

$$(2)\ p_{21} + p_{22} \geq 1$$

$$(3)\ p_{31} + p_{32} \geq 1$$

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$$(5) + (6) = (7)\ 3 \geq 4$$

Human vs Solver, Complexity Theory vs Modeling

In theory, the input must be the same when talking about complexity

- requires, e.g., input in CNF for comparing resolution vs cutting-planes
- does not allow efficient encodings which rely on the addition of new variables
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In practice, the way the constraints are expressed matters:

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In practice, current PB solvers behave as (slow) SAT solvers when given a CNF formula as input

Fitting Cutting-Planes into the CDCL architecture

Cutting Planes and Generalized Resolution

Many PB solvers have been designed based on the Generalized Resolution (Hooker, 1988).

$$\frac{\alpha\ell + \sum_{i=1}^{n} \alpha_i \ell_i \geq \delta_1 \qquad \beta\overline{\ell} + \sum_{i=1}^{n} \beta_i \ell_i \geq \delta_2}{\sum_{i=1}^{n} (\beta\alpha_i + \alpha\beta_i)\ell_i \geq \beta\delta_1 + \alpha\delta_2 - \alpha\beta} \text{ (cancellation)}$$

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As with the resolution rule in classical SAT solvers, these two rules can be used to learn new constraints during conflict analysis

Analyzing Conflicts

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This conflict is analyzed by applying the cancellation rule as follows:

$$\frac{6\bar{b} + 6c + 4e + f + g + h \geq 7 \qquad 5a + 4b + c + d \geq 6}{15a + 15c + 8e + 3d + 2f + 2g + 2h \geq 20}$$

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The constraint we obtain here is no longer conflicting!

Weakening

To preserve the conflict, the weakening rule must be used:

$$\frac{\alpha\ell + \sum_{i=1}^{n} \alpha_i \ell_i \geq \delta}{\sum_{i=1}^{n} \alpha_i \ell_i \geq \delta - \alpha} \text{ (weakening)}$$

$$\frac{\alpha\ell + \sum_{i=1}^{n} \alpha_i \ell_i \geq d \qquad k \in \mathbb{N} \qquad 0 < k \leq \alpha}{(\alpha - k)\ell + \sum_{i=1}^{n} \alpha_i \ell_i \geq \delta - k} \text{ (partial weakening)}$$

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Weakening can be applied in many different ways (Le Berre et al., 2020b)

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Another solution is to take advantage of the following property:

As soon as the coefficient of the literal to cancel is equal to 1 in at least one of the constraints, the derived constraint is guaranteed to be conflicting (Dixon, 2004)

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$$\dfrac{\dfrac{3\bar{a} + 3\bar{b} + c + d + e \geq 6}{3\bar{b} + c \geq 6 - 3 - 1 - 1 = 1}}{\bar{b} + c \geq 1}$$

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This strategy is equivalent to that used by solvers such as *SATIRE* (Whittemore and Sakallah, 2001) or *Sat4j-Resolution* to lazily infer clauses to use resolution based reasoning

Weakening and Division

In *RoundingSat* (Elffers and Nordström, 2018), the coefficient is rounded to one thanks to the division rule, applied after having weakened away some unfalsified literals

$$\frac{\sum_{i=1}^{n} \alpha_i \ell_i \geq \delta \qquad \rho \in \mathbb{N}^*}{\sum_{i=1}^{n} \lceil \frac{\alpha_i}{\rho} \rceil \ell_i \geq \lceil \frac{\delta}{\rho} \rceil} \text{(division)}$$

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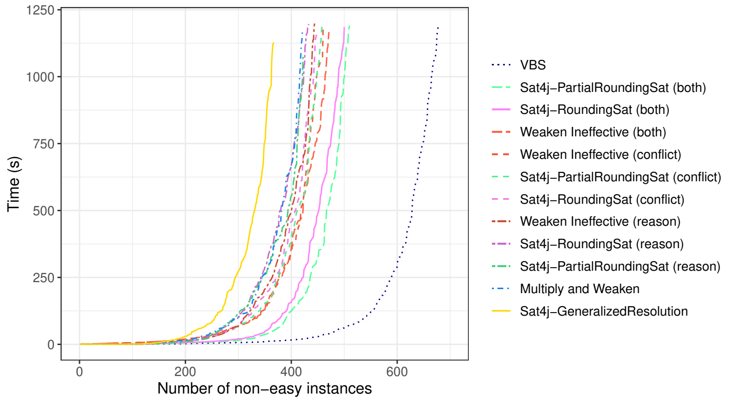
It is also possible to apply partial weakening before division to infer stronger constraints

Many Different Strategies



When to Stop Conflict Analysis?

For clausal analysis:

- stop when a single literal from current decision level remains
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For PB analysis, no such syntactical detection:

- depends on the weights of the literals assigned at each decision level
- backjump at the first decision level propagating a truth value

An Achilles Heel in the Cutting Planes Proof System

Irrelevant Literals (Le Berre et al., 2020a)

Cutting planes rules may introduce irrelevant literals

$$\frac{3d + a + b + c \geq 3 \qquad 3\bar{d} + 2a + 2b \geq 3}{3a + 3b + c \geq 3}$$

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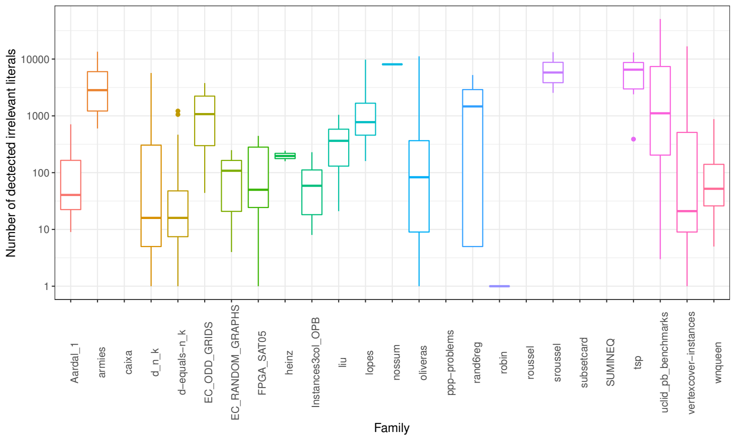
Production of Irrelevant Literals



Statistics about the production of irrelevant literals in *Sat4j-GeneralizedResolution* for each family of benchmarks (logarithmic scale)

Artificially Relevant Literals

Irrelevant literals may become artificially relevant, in which case they may impact the strength of the derived constraints

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Detecting irrelevant literals is NP-hard, we thus introduce an incomplete algorithm for removing them

Detecting Irrelevant Literals (1)

Irrelevant literals can be detected thanks to this reduction to subset-sum

$$\ell \text{ is irrelevant in } \alpha\ell + \sum_{i=1}^{n} \alpha_i \ell_i \geq \delta$$

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A dynamic programming algorithm can decide whether any of the equalities has a solution in pseudo-polynomial time with a single run

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Our algorithm solves subset-sum modulo a fixed number, or even several numbers

Removing Irrelevant Literals

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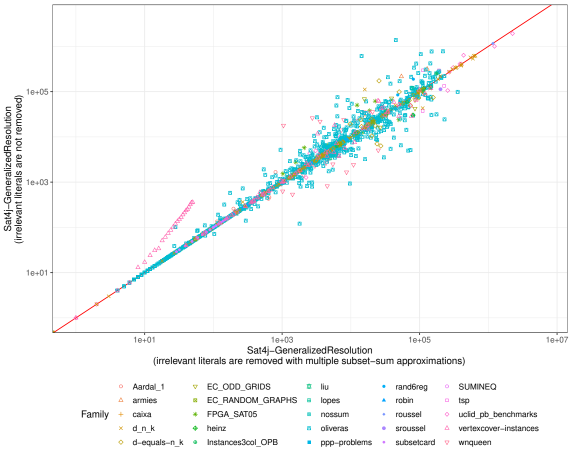
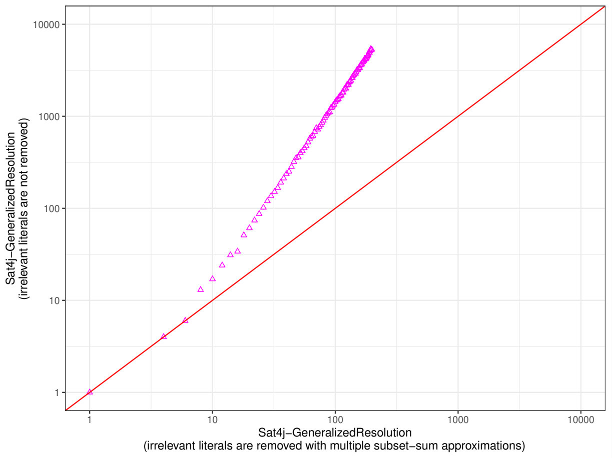
In practice, we use a heuristic to decide which strategy to apply, as none of them is better than the other

Impact of the Removal of Irrelevant Literals on the Proof



Comparison of the size of the proofs (number of cancellations) built by *Sat4j-GeneralizedResolution* with and without the removal of irrelevant literals on all benchmarks (logarithmic scale)

Focus on the Vertex-Cover Family: Experimental Results



Comparison of the size of the proofs (number of cancellations) built by *Sat4j-GeneralizedResolution* with and without the removal of irrelevant literals on vertex-cover instances (logarithmic scale)

Focus on the Vertex-Cover Family: *Sat4j*'s Behavior

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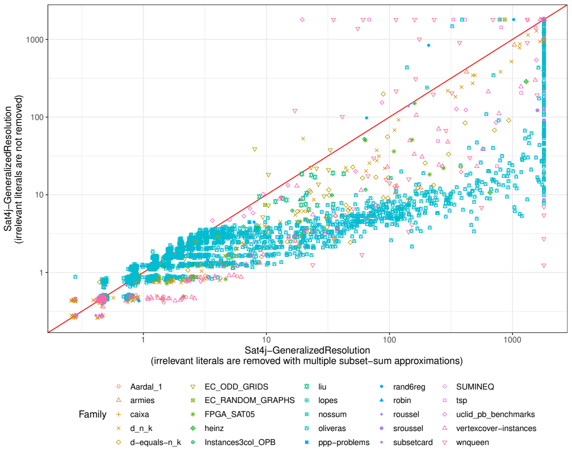
Even few irrelevant literals can lead to the production of an exponentially larger proof

Impact of the Removal of Irrelevant Literals on the Runtime



Comparison of the runtime of *Sat4j-GeneralizedResolution* with and without the removal of irrelevant literals on all benchmarks (logarithmic scale)

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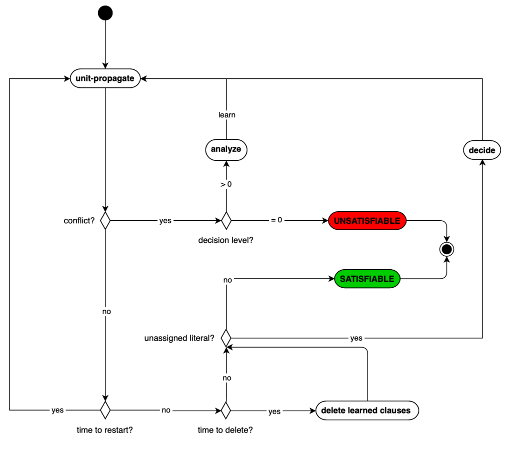
In the context of the current partial assignment, it is easy to detect ineffective literals, but they can only be weakened away (as ineffective literals may be relevant)

Adapting further PB Solvers to CDCL

CDCL Architecture Recap



Overview of the CDCL Algorithm

Motivation

It is well known that, in addition to conflict analysis, several features of SAT solvers are crucial for solving problems efficiently, such as:

- branching heuristic
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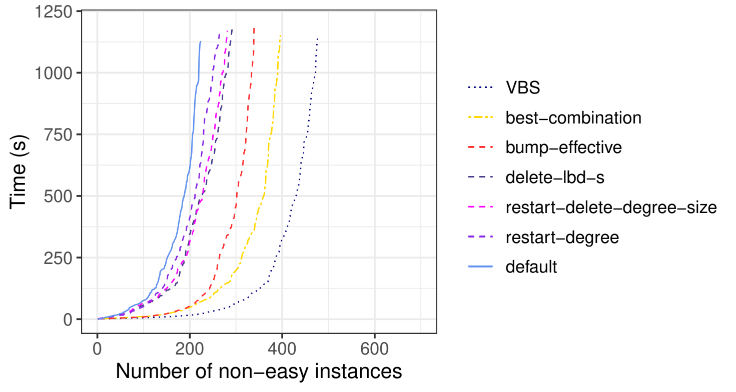
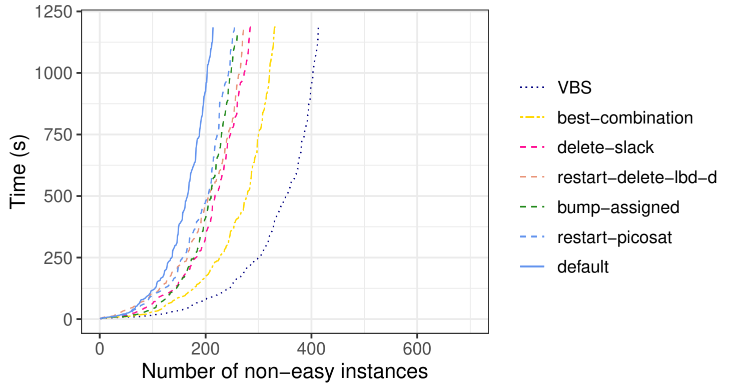
These features are mostly reused as is by current PB solvers, without taking into account the particular properties of PB constraints

Our main finding is that considering the size of the coefficients and the current partial assignment allows to significantly improve the solver

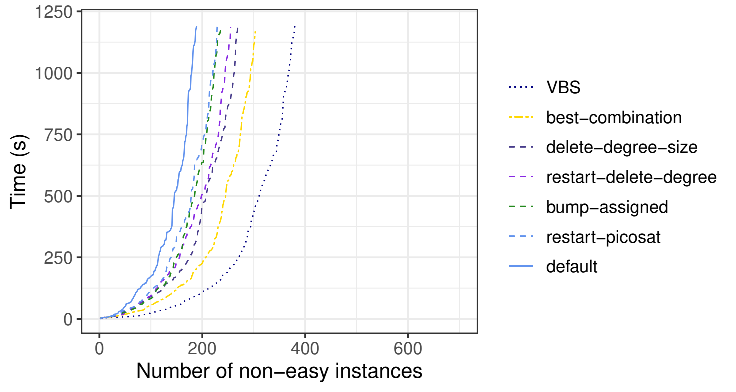
Experimental Results (Sat4j-GeneralizedResolution)



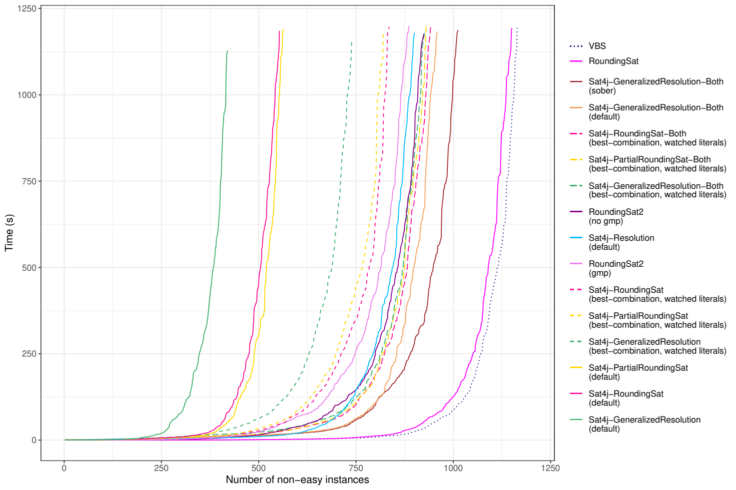
Experimental Results (Sat4j-RoundingSat)



Experimental Results (Sat4j-PartialRoundingSat)



Comparison of Sat4j with RoundingSat



Deeper Dive into Sat4j

Leveraging Properties of PB Constraints for Fine Tuning Sat4j

Let us consider again a confict analysis

$$3\bar{a}(?@?) + 3\bar{f}(?@?) + d(?@?) + e(?@?) \geq 5$$

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We now apply the cancellation rule between these two constraints:

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The PB constraints involved in this conflict analysis have very different properties compared to clauses!

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This means that the scores of the variables a, f, d and e are incremented

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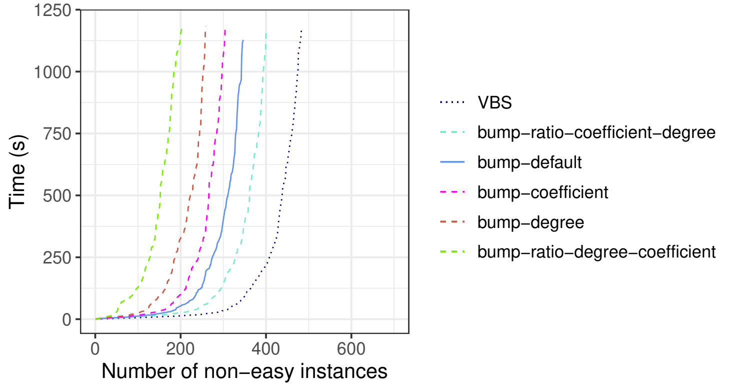
- `bump-ratio`: the score of each variable is incremented by the ratio of the degree by their coefficient in the constraint ($\frac{5}{3}$ for a and f)

(E)VSIDS for Making Decisions: Experiments



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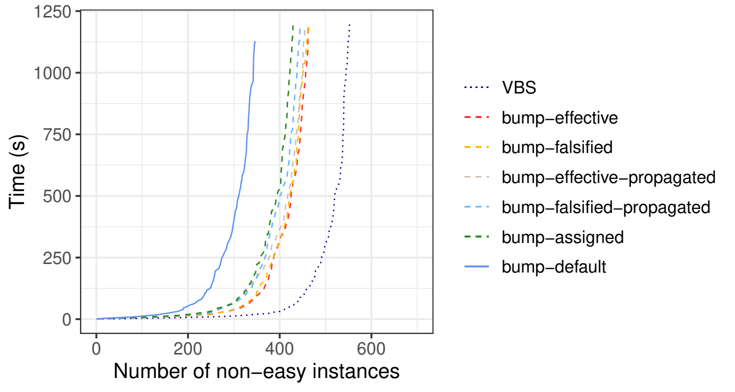

Quality of Learned Constraints: Classical Implementations

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Quality of Learned Constraints: Size and Coefficients

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We thus introduce 5 new definitions of LBD:

- `lbd-a`: the LBD is computed over assigned literals only
- `lbd-s`: the LBD is computed over assigned literals, and unassigned literals are considered assigned at the same (dummy) decision level
- `lbd-d`: the LBD is computed over assigned literals, and unassigned literals are considered assigned at different (dummy) decision levels
- `lbd-f`: the LBD is computed over falsified literals only
- `lbd-e`: the LBD is computed over effective literals only

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We thus introduce the following deletion strategies, based on the different quality measures we presented:

- `delete-degree`
- `delete-degree-size`
- `delete-lbd-a`
- `delete-lbd-s`
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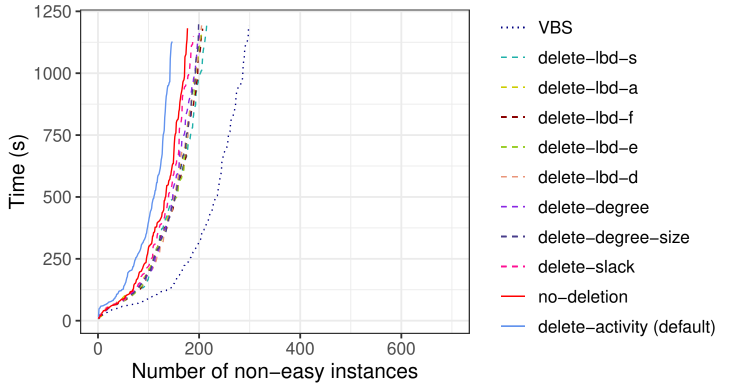

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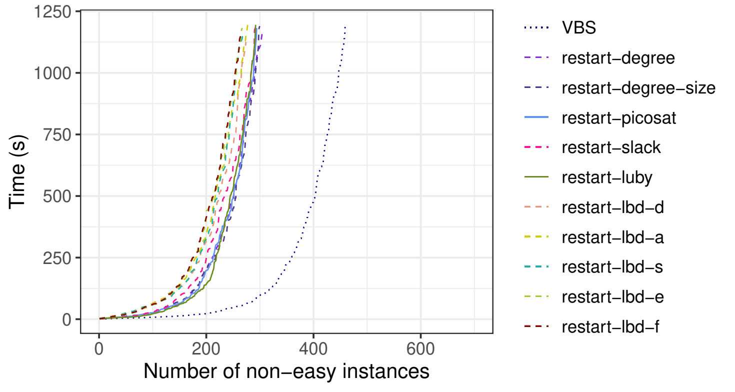
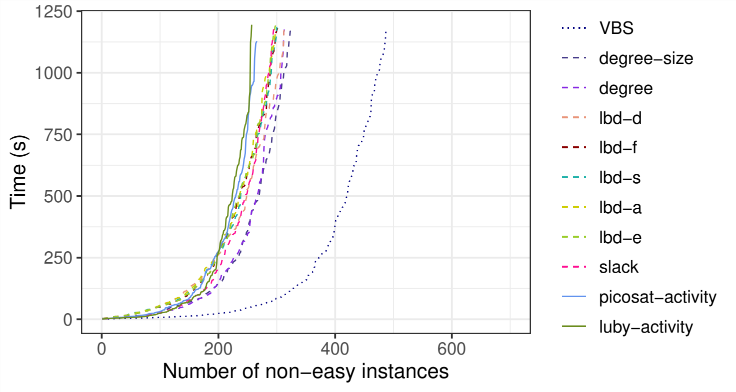


Quality of Learned Constraints: Experiments (Del. + Restarts)



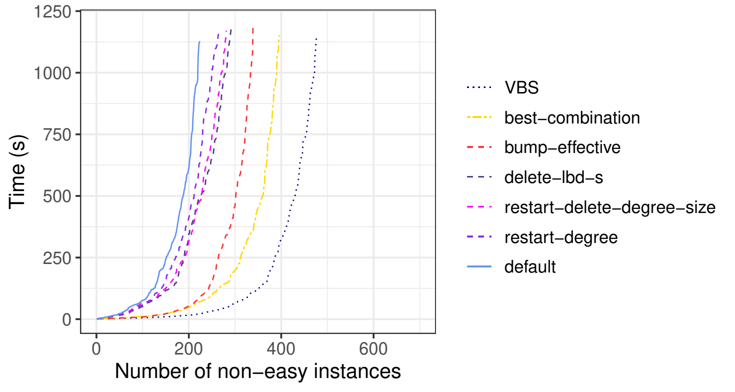
Putting Things Together: Brief Recap

For the moment, we have observed that the following individual strategies have the most important impact on the performance of the solver:

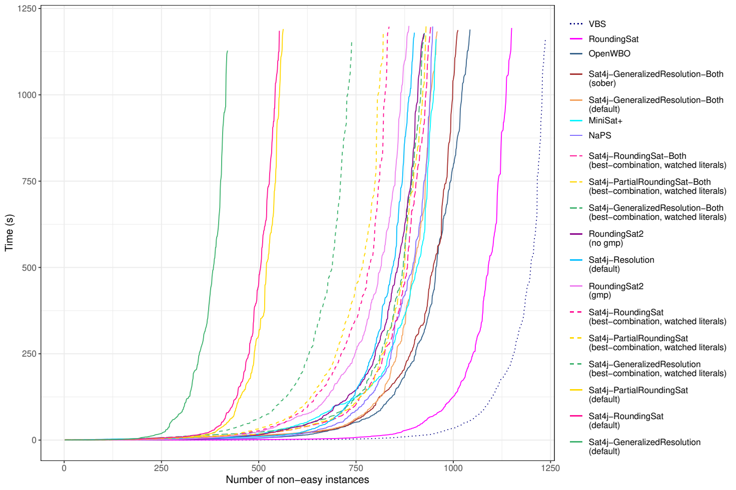
- `bump-effective`
- `delete-lbd-s`
- `restart-degree`

Let us combine all these strategies!

Putting Things Together: Experiments



Comparison of Sat4j with State-of-the-Art Solvers



Recovering Cardinality Constraints

Semantic cardinality detection

- Theory tells us that Cutting Planes should work on CNF
- Current implementations do not
- Can we find a way to help PB solvers work on CNF?
- Caution: we need a general process, not one dedicated to a given problem or constraint

Example of a Human Proof ($n = 3$) again

$$(1)\ p_{11} + p_{12} \geq 1 \ (2)\ p_{21} + p_{22} \geq 1 \ (3)\ p_{31} + p_{32} \geq 1$$

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PHP: cardinality constraints *vs.* clauses

PHP: inconsistency proof computation time

- pigeons-100-hole.cnf:
  - resolution $\rightarrow$ timeout (900s)
  - generalized resolution (Hooker, 1988) $\rightarrow$ timeout (900s)
- pigeons-100-hole.opb:
  - resolution $\rightarrow$ timeout (900s)
  - generalized resolution (Hooker, 1988) $\rightarrow < 1s$.

Representation of constraints matters

Cardinality constraints *vs.* clauses

- pros :
  - a cardinality constraint may replace an exponential number of clauses or prevent the use of auxiliary variables
  - allow to use strong proof systems (generalized resolution, cutting planes)
- cons:
  - difficult detection : many encoding exist to translate cardinality constraints into CNF
  - deriving cardinality constraints using Cutting Planes proof system does not fit well with CDCL architecture

Some known encodings

Short list of known encodings :

- Pairwise encoding (Cook et al., 1987)
- Nested encoding
- Two product encoding (Chen, 2010)
- Sequential encoding (Sinz, 2005)
- Commander encoding (Frisch and Giannaros, 2010)
- Ladder encoding (Gent and Nightingale, 2004)
- Adder encoding (Eén and Sörensson, 2006)
- Cardinality Networks (Asín et al., 2009)
- ...

Syntactic *vs.* semantic detection

- Syntactic detection:
  - need of an *ad hoc* algorithm for each {encoding,k}
- Our semantic detection (Biere et al., 2014)
  - based on unit propagation
  - adapted to any encoding preserving arc-consistency
  - may potentially detect constraints that were not known at encoding time
  - detection may be altered by auxiliary variables
- More recent: inprocessing detection (Elffers and Nordström, 2020)

Semantic detection of at-most-k constraint

detecting a cardinality constraint in a semantic way:

1. select a clause of size n, and translate it into an AtMost-k of degree $n-1$:
$$\bigvee_{i=1}^{n} x_i \ \equiv \ \sum_{i=1}^{n} \neg x_i \leq n-1$$
2. look for literals m_j to extend this constraint:
$$\sum_{i=1}^{n} (\neg x_i) + m_1 + ... + m_p \leq n-1$$

Semantic detection of at-most-k constraint: example

formula :

$$\begin{aligned} \neg x_1 &\vee \neg x_2 \\ \neg x_1 &\vee \neg x_4 \\ x_4 &\vee \neg x_3 \\ \neg x_2 &\vee \neg x_5 \\ x_5 &\vee \neg x_3 \end{aligned}$$

detection of $\sum_{i=1}^{3} x_i \leq 1$

Semantic detection of at-most-k constraint: example

$\neg x_1 \vee \neg x_2$

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Semantic detection of at-most-k constraint: example

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$\neg x_2 \vee \neg x_5$

$x_5 \vee \neg x_3$

$$\neg x_1 \vee \neg x_2$$
$$\equiv$$
$$x_1 + x_2 \leq 1$$

detection of $\sum_{i=1}^{3} x_i \leq 1$

Semantic detection of at-most-k constraint: example

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$$\neg x_1 \vee \neg x_2 \equiv x_1 + x_2 \leq 1$$

detection of $\sum_{i=1}^{3} x_i \leq 1$

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formula :

$\neg x_1 \vee \neg x_2$

$\neg x_1 \vee \neg x_4$

$x_4 \vee \neg x_3$

$\neg x_2 \vee \neg x_5$

$x_5 \vee \neg x_3$

$$\neg x_1 \vee \neg x_2$$
$$\equiv$$
$$x_1 + x_2 \leq 1$$

$$\text{PU}(x_1) = \{\ x_1, \neg x_2, \neg x_3, \neg x_4 \quad \}$$

$$\text{detection of } \sum_{i=1}^{3} x_i \leq 1$$

Semantic detection of at-most-k constraint: example

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$$x_1 + x_2 + x_3 \leq 1$$

detection of $\sum_{i=1}^{3} x_i \leq 1$

Cardinality constraint extension

Cardinality constraint extension:

1. let $\alpha = \sum_{i=1}^{n} x_i \leq k$
2. initialization of the propagation set $\gamma = \{v_i, \neg v_i \mid v \in \mathrm{PS}\}$
3. for each subset of k literals x_i, we compute the unit propagation set δ, and we refine the propagation set:

$$\gamma \leftarrow \gamma \cap \delta$$

4. if there exists $m \in \gamma$, then $\alpha = \sum_{i=1}^{n} x_i + \neg m \leq k$ and goto 2

Experimental evaluation

- aim of the experiments: check that detected constraints help a generalized resolution based solver
- solvers:
  - Lingeling: able to detect pairwise encoding
  - Synt.+Sat4jCP, Sem.+Sat4jCP, Sat4jCP w/o preprocessing
  - SBSAT: able to detection cardinality constraints *via* compilation steps
- Intel Xeon@2.66GHz, 32Go RAM, timeouts=900s

Results

Influence of detected constraints for some encodings of PHP:

Preprocessing Solver	#inst.	Lingeling Lingeling	Synt.(Riss) Sat4jCP	Sem.(Riss) Sat4jCP	∅ SBSAT	∅ Sat4jCP
Pairwise	14	14 (3s)	13 (244s)	14 (583s)	6 (0s)	1 (196s)
Binary	14	3 (398s)	2 (554s)	7 (6s)	6 (7s)	2 (645s)
Sequential	14	0 (0s)	14 (50s)	14 (40s)	10 (6s)	1 (37s)
Product	14	0 (0s)	14 (544s)	11 (69s)	6 (25s)	2 (346s)
Commander	14	1 (3s)	7 (0s)	14 (40s)	9 (187s)	1 (684s)
Ladder	14	0 (0s)	11 (505s)	11 (1229s)	12 (26s)	1 (36s)

solved instances (computation time of solved instances)

Results

Influence of detected constraints for *balanced block design* instances:

Preprocessing Solver	#inst.	Lingeling Lingeling	Synt.(Riss) Sat4jCP	Sem.(Riss) Sat4jCP	∅ SBSAT	∅ Sat4jCP
Sgen unsat	13	0 (0s)	13 (0s)	13 (0s)	9 (614s)	4 (126s)
Fixed bandwidth	23	2 (341s)	23 (0s)	23 (0s)	23 (1s)	13 (1800s)
Rand. orderings	168	16 (897s)	168 (7s)	168 (8s)	99 (2798s)	69 (3541s)
Rand. 4-reg.	126	6 (1626s)	126 (4s)	126 (5s)	84 (2172s)	49 (3754s)

solved instances (computation time of solved instances)

Further Results

- “crossed” constraints: Sudoku grid
  - Sudoku 9x9: syntactic preprocessing detects 300/324 constraints, semantic one detects 324/324 constraints
  - Sudoku 16x16: syntactic preprocessing detects 980/1024 constraints, semantic one detects 1024/1024 constraints

- Challenge benchmark of (Van Gelder and Spence, 2010) (clasp unable to solve within 24h): solved within a second thanks to semantic preprocessing (AtMost-3 constraints inside)

Some Open Questions about PB Conflict Analysis

#1 - Improving the Backjump Level

The backjump level computed based on the 1-UIP scheme is not optimal in general in PB solvers

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\boxplus & 3i(0@2) + 3j(0@4) + \underline{2\bar{f}(1@4)} + \underline{2\bar{g}(1@4)} + h(1@4) \geq 5 \\
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\end{array}$$

Continuing conflict analysis after the production of an assertive constraint may allow to improve the backjump level, but it is not always true

#2 - Improving Conflict Detection

Consider these 4 constraints:

$$\chi_1 \equiv a + \bar{b} + \bar{c} \geq 2$$
$$\chi_2 \equiv 3b + 3d + e + f \geq 4$$
$$\chi_3 \equiv 2c + \bar{e} + \bar{f} \geq 2$$
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If we choose χ_4 as conflict to analyze, we have:

$$\cfrac{\cfrac{\cfrac{\chi_4 \qquad \chi_3}{b + c + \bar{d} \geq 1} \qquad \chi_2}{4b + 3c + e + f \geq 4} \qquad \chi_1}{3a + b + e + f \geq 4}$$

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If we choose χ_2 as conflict to analyze, we have:

$$\dfrac{\dfrac{\chi_2 \qquad \chi_3}{3b + 3d + 2c \geq 4} \qquad \chi_1}{3d + 2a + b \geq 4}$$

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If we choose χ_2 as conflict to analyze, we have:

$$\cfrac{\cfrac{\chi_2 \qquad \chi_3}{3b + 3d + 2c \geq 4} \qquad \chi_1}{3d + 2a + b \geq 4}$$

None of the two constraints is stronger than the other

Another Perspective on Normalized Forms

Why do we need a *normalized* form?

Recall that all constraints we consider are of the form:

$$\sum_{i=1}^{n} \alpha_i \ell_i \geq \delta$$

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This normalized form is needed to apply the cancellation rule, because we need literals to be in the same side of the relational operator:

$$\frac{\alpha\ell + \sum_{i=1}^{n} \alpha_i \ell_i \geq \delta_1 \qquad \beta\overline{\ell} + \sum_{i=1}^{n} \beta_i \ell_i \geq \delta_2}{\sum_{i=1}^{n} (\beta\alpha_i + \alpha\beta_i)\ell_i \geq \beta\delta_1 + \alpha\delta_2 - \alpha\beta} \text{ (cancellation)}$$

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So, why don't we use a $\leq$-based normalized form?

Why do we prefer $\geq$ over $\leq$?

The main advantage of using $\geq$ is that it allows to easily represent clauses

$$\ell_1 \vee \ldots \vee \ell_n \equiv \ell_1 + \ldots + \ell_n \geq 1$$

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Here, the clause above would be written as follows in a $\leq$-based normalized form

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However, this normalized form is not always practical...

Normalized Form and PB Optimization

On optimization problems, for instance, the solver is asked to minimize an objective function of the form:

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If the solver finds a model of the formula which makes the objective function equal to a value o, then the following constraint is added to the solver:

$$\sum_{i=1}^{n} \alpha_i \ell_i \leq o - 1 \equiv \sum_{i=1}^{n} \alpha_i \bar{\ell}_i \geq \sum_{i=1}^{n} \alpha_i - o + 1$$

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If the coefficients α_i are big, the degree of the constraint above will become even bigger, even if the value o is small

The Role of the Saturation Rule

Recall that PB solver can use the saturation rule

$$\frac{\sum_{i=1}^{n} \alpha_i \ell_i \geq \delta}{\sum_{i=1}^{n} \min(\alpha_i, \delta) \ell_i \geq \delta} \text{ (saturation)}$$

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This rule makes the degree of a constraint an upper bound of its coefficients, but if the degree becomes big, then it may allow coefficients to grow while applying consecutive cancellation step

Another Rule for $\leq$ Constraints?

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To this end, we may borrow the following rule used in MIP preprocessing

$$\frac{\alpha\ell + \sum_{i=1}^{n} \alpha_i \ell_i \leq \delta \qquad \sum_{i=1}^{n} \alpha_i \leq \delta}{\left(\alpha + \sum_{i=1}^{n} \alpha_i - \delta\right) \ell + \sum_{i=1}^{n} \alpha_i \ell_i \leq \sum_{i=1}^{n} \alpha_i}$$

Recap on Normalized Constraints

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- However, the $\leq$ is sometimes preferable, e.g., on optimization problems
- All rules used by PB solvers are not always applicable to both representations, requiring to adapt the proof system to the representation

Ideally, we could choose either the $\geq$ or the $\leq$ representation depending on which is "better", as this is done in encodings for instance

Conclusion and Perspectives

Conclusion

- Implementations of the cutting planes proof system in PB solvers are not fully satisfactory, as its strength is not fully exploited
- Irrelevant literals may be produced during conflict analysis, and lead to the inference of weaker constraints
- Applying the weakening rule on ineffective literals is a possible (aggressive) counter-measure
- Applying partial weakening and division gives better performance

Conclusion

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- Irrelevant literals may be produced during conflict analysis, and lead to the inference of weaker constraints
- Applying the weakening rule on ineffective literals is a possible (aggressive) counter-measure
- Applying partial weakening and division gives better performance

- Complementary heuristics implemented in CDCL PB solvers can be adapted to take into account properties of PB constraints and to improve the performance of *Sat4j*

Perspectives

- Find other strategies for applying cutting planes rules so as to exploit more power of this proof system
- Design such strategies so as to prevent the production of irrelevant literals instead of removing them
- Combine the weakening strategies to exploit their complementarity
- Identify possible interactions between the new heuristics

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- Consider their impact on the resolution of optimization problems

- Improve the detection of the optimal backjump level during conflict analysis

- Improve the detection of conflicts to deal with the conflictual reasons encountered during conflict analysis

Deep Dive into CDCL Pseudo-Boolean Solvers

focusing on the implementation in *Sat4j*

Daniel Le Berre[1], Romain Wallon[2]

Februrary 23rd, 2021

[1]CRIL, Univ Artois & CNRS
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