Inconsistency Measurement Thanks to MUS Decomposition

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ABSTRACT

Bearing contradictory knowledge is often unavoidable among multi-agents. Measuring inconsistency degrees of knowledge bases of different agents facilitates the understanding of an agent to her environment. Several semantics or syntax-based approaches have been proposed to quantify inconsistencies. In this paper, we propose a new inconsistency measuring framework based on both minimal unsatisfiable sets and maximal consistent sets. Firstly, we define a graph representation of knowledge bases, based on which we furthermore explore the logical property of the *Additivity* condition. Then, we show how the structure of the proposed graph representation can be used to discriminate, in a fine-grained way, the responsibility of each formula or a set of formulae for the inconsistency of a knowledge base. Finally, we extend our framework to provide an inconsistency measure for a whole knowledge base. All the proposed measures are shown satisfying the desired properties.

Categories and Subject Descriptors

H.3 [**Agent Reasoning**]: Knowledge representation

General Terms

Theory

Keywords

Measuring Inconsistency, Classical Logic

1. INTRODUCTION

Measuring inconsistency has proven useful and attractive in diverse scenarios including software specifications [21], e-commerce protocols [5], belief merging [28], news reports [13], integrity constraints [8], requirements engineering [21], databases [22, 11], ontologies [33], semantic web [33], and network intrusion detection [23].

Indeed, we cannot expect large-sized knowledge bases free of inconsistency in real applications, such as multi-agents communicating with each other to build a common knowledge base or to perform certain actions in a complex environment. For illustration purpose, we consider an example simplified from the Multi-agent System ARTOR (ARTificial ORganizations [30]) which simulates the partnership of organizations – each organization owns agents responsible for purchasing and selling products. We consider a scenario where each product has N grades of price and M grades of quantity (higher grade, larger value, e.g. \$10 for the price grade 1 and \$20 for the grade 2). Selling agents wish to sell products at a price as expensive as possible and with a quantity as large as possible, but buying agents wish as cheap as possible and have quantity upper bounds. By $p_i (1 \leq i \leq N)$ and $q_j (1 \leq j \leq M)$, we denote a trade of price grade i and of quantity grade j, respectively. Consider the following two agents:

$$\begin{array}{rcll} \text{Buying agent :} & & \text{Selling agent :} & \\ p_{i+1} \to p_i & | & p_i \to p_{i+1} & (1) \\ q_{j+1} \to q_j & | & q_j \to p_{j+1} & (2) \\ p_1 \land p_2 \land p_3 \land \neg p_4 & | & \neg p_5 \land p_6 \land \cdots \land p_N & (3) \\ q_1 \land q_2 \land \neg q_3 & | & \neg q_6 \land q_7 \land \cdots \land q_M & (4) \end{array}$$

Item 1 (resp. 2) says that if the buying agent accepts the price (resp. quantity) at grade $i+1$ (resp. $j+1$), then he accepts lower prices (resp. less quantities); Whilst the selling agent accepts the price (resp. quantity) at grade i (resp. j), then he accepts higher prices (resp. larger quantities). Together with Item 3 (resp. 4), we can read that the buying agent agrees to pay at most at price grade 3 and with the maximal quantity grade 2, but the selling agent accepts minimal price grade 6 and quantity grade 7. It is easy to see that there are contradictions between these two agents (e.g. $\{p_3, \neg q_7\}$ for the buying agent but $\{\neg p_3, q_7\}$ for the selling agent). To make the trade successful, agents should be equipped with the ability of reasoning with inconsistency. Moreover, if an agent knows the different inconsistency degrees between him and others, a better trade plan could be arranged when multiple sellers and buyers are available.

Analyzing conflicting information has gained a considerable attention and become an important issue in computer science recently [2]. Indeed, measuring inconsistency is helpful to compare different knowledge bases and to evaluate their quality [7]. Consider again the above example, giving the opportunity for a buyer agent to choose among different selling agents, naturally he may try to choose the one that is less inconsistent.

To understand the nature of inconsistency and to quantify it in turn, a number of logic-based inconsistency measures have been studied, including the maximal η-consistency [18], measures based on variables or via multi-valued models [7,

Appears in: *Proceedings of the 13th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2014), Lomuscio, Scerri, Bazzan, Huhns (eds.), May, 5–9, 2014, Paris, France.*

12, 27, 13, 9, 20, 31, 19], n-consistency and n-probability [6], minimal inconsistent subsets based inconsistency measures [15, 24, 25, 32], the Shapley inconsistency value [14, 16], and the inconsistency measurement based on minimal proofs [17]. Although it is hardly possible to have a complete comparison of the proposed measures. One way to categorize the existing measures can be based on the dependence of syntax or semantics: Semantics based ones aim to compute the proportion of the language affected by inconsistency. The inconsistency values belonging to this class are often based on some paraconsistent semantics and syntax independent. Whilst, syntax based ones are concerned with the minimal number of formulae that cause inconsistencies. Viewing Minimal Unsatisfiable Subsets (MUSes) as the cornerstones of inconsistency, it is natural to derive inconsistency measures from MUSes of a knowledge base. Another possible classification of different measures originates in the measuring objective: formula oriented or knowledge base oriented. For example, the inconsistency measures in [14, 16] are to measure the contribution of a formula to the inconsistency of a whole knowledge base that it belongs to, while the rest measures mentioned above are to measure the inconsistency degree of a whole base. Some basic properties [16], such as *Consistency, Monotony, Free Formula Independence, and MinInc*, are also proposed to qualify inconsistency measures.

In this paper, we propose a new approach for measuring inconsistency, both formula oriented and knowledge base oriented. It is inspired, on the one hand, by the observation that existing measures fail to distinguish certain knowledge bases which should have different inconsistency degrees. On the other hand, a specific property is explored, namely *Additivity* that is rarely discussed in the literature due to its modeling difficulty [16]. Our study is based on a novel analysis of connections among MUSes, which is shown useful and general to quantify more finely the inconsistency responsibility of a formula and the inconsistency degree of a whole base. Note that this is unlike the oriented links between MUSes explored in [1] which says that the resolution of one MUS allows for the resolution of the other. By looking inside MUSes and taking into account the correlations between them, the proposed analysis of MUSes structure lead to several interesting measures different from the existing ones. Moreover, we propose an *enhanced additivity* property and show that our measures satisfy such a property as well as several basic properties.

The paper is organized as follows: Section 2 reviews approaches to measuring inconsistencies based on minimal inconsistent subsets and maximal consistent subsets. In Section 3, we revisit the additivity property and propose the graph representation of a knowledge base by which we introduce the notion of MUS-decomposition. This notion is then used in Section 4 to evaluate the degree of inconsistency of each formula in the knowledge base. In Section 5, we generalize our framework to quantify inconsistency of a whole base. Section 6 concludes the paper and discusses some perspectives.

2. PRELIMINARIES

Through this paper, we consider the propositional language $\mathcal{L}$ built from a finite set of propositional symbols $\mathcal{P}$ under connectives $\{\neg, \wedge, \vee, \rightarrow\}$. We will use $a, b, c, \dots$ to denote propositional variables, and Greek letters $\alpha, \beta, \gamma, \dots$ for formulae. The symbol $\perp$ denotes contradiction. For a set S, we denote $|S|$ its cardinality.

A knowledge base K consists of a finite set of propositional formulae. K is inconsistent if there is a formula α such that $K \vdash \alpha$, and $K \vdash \neg\alpha$, where $\vdash$ is the deduction in classical propositional logic. If K is inconsistent, then one can define the notion of *Minimal Unsatisfiable Subset* (MUS) as an unsatisfiable set of formulae $\mathcal{M}$ in K such that any of its subsets is satisfiable. Formally,

Definition 1 (MUS). *Let K be a knowledge base and $\mathcal{M} \subseteq K$. $\mathcal{M}$ is a minimal unsatisfiable (inconsistent) subset of K iff $\mathcal{M} \vdash \perp$ and $\forall \mathcal{M}' \subset \mathcal{M}$, $\mathcal{M}' \nvdash \perp$.*

Clearly, an inconsistent knowledge base K can have multiple minimal inconsistent subsets. The set of minimal inconsistent subsets of K is denoted as $MUSes(K) = \{\mathcal{M} \subseteq K \mid \mathcal{M}\ is\ a\ MUS\}$. When a MUS is singleton, the single formula in it is called a *self-contradictory formula*. A formula α not involved in any MUS of K is called a *free formula*, meaning that α does not have any relationship with the inconsistency of K. Formally, $SelfC(K) = \{\alpha \in K \mid \{\alpha\} \vdash \perp\}$, and $free(K) = \{\alpha \mid \text{ for all } \mathcal{M} \in MUSes(K), \alpha \notin \mathcal{M}\}$.

At the same time, we can define the *Maximal Satisfiable Subset* (MSS), and Hitting set as follows:

Definition 2 (MSS). *Let K be a knowledge base and $\mathcal{M}$ a subset of K. $\mathcal{M}$ is a maximal satisfiable (consistent) subset of K iff $\mathcal{M} \nvdash \perp$ and $\forall \alpha \in K \setminus \mathcal{M}$, $\mathcal{M} \cup \{\alpha\} \vdash \perp$.*

We denote by $MSSes(K)$ the set of all maximal consistent subsets of K.

Definition 3 (Hitting set). *Given a universe U of elements and a collection $\mathcal{S}$ of subsets of U, $H \subseteq U$ is a hitting set of $\mathcal{S}$ if $\forall E \in \mathcal{S}, H \cap E \neq \emptyset$. We say that H is a minimal hitting set of $\mathcal{S}$ if H is a hitting set of $\mathcal{S}$ and each $H' \subset H$ is not a hitting set of $\mathcal{S}$.*

We denote by $LB_{HS}(K)$ the smallest size of a hitting set of K, i.e., $LB_{HS}(K) = min\{|H| \mid \text{H is a hitting set of } K\}$.

2.1 Inconsistency Measures

In this section, we review some inconsistency measures important and related to the rest of this paper.

There have been several contributions for measuring inconsistency in knowledge bases defined through minimal inconsistent subsets theories. In [16], Hunter and Konieczny introduce a scoring function allowing to measure the degree of inconsistency of a subset of a knowledge base. In details, for a subset $K' \subseteq K$, the scoring function is defined as the diminution of the number of minimal inconsistent subsets while K' is removed, i.e., $|MUSes(K)| - |MUSes(K \setminus K')|$. By extending the scoring function, the authors introduce an inconsistency measure of a whole base, defined as the number of minimal inconsistent subsets of K. Formally, $I_{MI}(K) = |MUSes(K)|$. In the same paper, a family of "MinInc inconsistency values" MIV based on minimal inconsistent subsets is also presented:

- $MIV_D(K, \alpha)$ is a simple measure that values 1 if α belongs to a minimal inconsistent subset and 0 otherwise.
- $MIV_\#$ is defined by the scoring function: $MIV_\#(K, \alpha) = |\{\mathcal{M} \in MUSes(K) \mid \alpha \in \mathcal{M}\}|$.

- MIV_C takes into account the size of each minimal inconsistent subset in addition to the number of $MUSes$ of K, formally $MIV_C(K,\alpha) = \sum_{\alpha\in\mathcal{M}|\mathcal{M}\in MUSes(K)} \frac{1}{|\mathcal{M}|}$.

Combining both minimal inconsistent subsets and maximal consistent subsets, Mu et al. present in [26] an approach to quantify the degree of inconsistency of a knowledge base. Another inconsistency value, called I_M measure, that deals with that combination, has been introduced in [10]. The I_M measure counts for a given knowledge base, the number of its $MSSes$ and its Self-contradictory formulae (subtraction of 1 is required to make $I_M(K) = 0$ when K is consistent):

$$I_M(K) = |MSSes(K)| + |SelfC(K)| - 1.$$

Recently, in [17] a new framework based on the notion of minimal proof is proposed to measure inconsistency degrees. This framework uses conflicting minimal proofs to characterize the inconsistency of any subset of formulae in a base.

3. MUS PARTITIONING IN KNOWLEDGE BASES

There are a set of well accepted basic properties that inconsistency measures should satisfy (see Definition 4), while leaving one property *Additivity* debatable [15]. In this section, we propose an enhancement of the additivity property to make it more intuitive and give a way to modify the I_M measure introduced above to satisfy the enhance additivity.

Definition 4 ([16]). *Let K and K' be two knowledge bases, α and β two formulae. A set of properties of an inconsistency measure I is defined as follows:*

(1) Consistency: $I(K) = 0$ iff K is consistent,

(2) Monotony: $I(K) \leq I(K \cup K')$,

(3) Free Formula Independence: if α is a free formula in $K \cup \{\alpha\}$, then $I(K \cup \{\alpha\}) = I(K)$,

(4) MinInc: If $M \in MUSes(K)$, then $I(M) = 1$.

The monotony property shows that the inconsistency value of a knowledge base has to be non-decreasing while adding new formulae. The free formula independence property states that the set of formulae not involved in any minimal inconsistent subset should have no contribution to inconsistency. And MinInc means that a single MUS as a whole has inconsistency value 1. Besides, another property called *Additivity*[1] has been proposed in [16].

Definition 5 (Additivity). *Let $K_1, \dots, K_n$ be knowledge bases and I an inconsistency measure. I is additive if it satisfies the following condition: If $MUSes(K_1 \cup \dots \cup K_n) = MUSes(K_1) \oplus \dots \oplus MUSes(K_n)$*[2]*, then $I(K_1 \cup \dots \cup K_n) = I(K_1) + \dots + I(K_n)$.*

The additivity was proposed to ensure that the amount of conflict of a knowledge base K can be obtained by summing up the degrees of its sub-bases K_i under the condition that $\{MUSes(K_i)\}$ is a partition of $MUSes(K)$. However, Luce and Raiffa have pointed out that the interaction of sub-bases (sub-games in [29]) is not taken into account by Additivity, which is one of the criticisms about this condition [29, 16]. Although the partitionability of $MUSes$ is used to describe a sort of interaction in Definition 5, we argue that it is not enough. Consider the following example:

Example 1. *Let $K_1 = \{a, \neg a\}, K_2 = \{\neg a, a \wedge b\}$, and $K_3 = \{c, \neg c\}$, each of which has only one single MUS. Consider two bases $K = K_1 \cup K_2$, and $K' = K_1 \cup K_3$. Clearly, $MUSes(K) = MUSes(K_1) \oplus MUSes(K_2)$, and $MUSes(K') = MUSes(K_1) \oplus MUSes(K_3)$. For any measure I, if I satisfies Additivity by Definition 5, we should have $I(K) = I(K_1)+I(K_2)$, and $I(K') = I(K_1)+I(K_3)$. If I furthermore satisifes MinInc (see Definition 4), single MUS leads to the same inconsistency value. Then $I(K) = I(K')$, which is counterintuitive because the components of $MUSes(K')$ are less interactive, thus more spreading, than that of $MUSes(K)$. And in turn K' should, intuitively, contains more inconsistencies than K.*

To enhance the consideration of interaction among sub-bases, we propose the following Enhanced Additivity:

Definition 6 (Enhanced Additivity). *Let $K_1, \dots, K_n$ be knowledge bases and I an inconsistency measure. If $MUSes(K_1 \cup \dots \cup K_n) = MUSes(K_1) \oplus \dots \oplus MUSes(K_n)$ and $\{\alpha \in \mathcal{M} \mid \mathcal{M} \in MUSes(K_i)\} \cap \{\beta \in \mathcal{M} \mid \mathcal{M} \in MUSes(K_j)\} = \emptyset$ for all $1 \leq i \neq j \leq n$, $I(K_1 \cup \dots \cup K_n) = I(K_1)+\dots+I(K_n)$. I is then called an independent-additive measure.*

Note that the enhanced additivity requires an extra precondition, which is to encode a stronger independence among sub-bases to perform additivity. The enhanced additivity can exclude the counterintuitive conclusion as given in Example 1: suppose I satisfies the enhanced additivity, then we have $I(K') = I(K_1)+I(K_3)$, but not necessarily $I(K') = I(K_1)+I(K_2)$. Hence $I(K)$ is not necessarily equal to $I(K')$.

Alternatively, the enhanced-additivity, given below, can be defined through multi-set partition instead of set partition as used in Definition 6.

Definition 7. *I is called multi-set additive if it satisfies: $I(K_1 \cup \dots \cup K_n) = \sum_{i=1}^{k} I(K_i)$ when $MUSes(\bigcup_{i=1}^{n} K_i) = \biguplus_{i=1}^{n} MUSes(K_i)$, where $\biguplus$ is the multi-set union over sets.*

For Example 1, whilst $MUSes(K) = \{\{a, \neg a\}, \{\neg a, a \wedge b\}\}$, we have $MUSes(K_1) \biguplus MUSes(K_2) = \{\{a, \neg a\}, \{\neg a, a \wedge b\}, \{a, \neg a\}\}$. This leads to the conclusion that it is unnecessarily to have $I(K) = I(K_1) + I(K_2)$, the same as using Definition 6. In fact, it is easy to see that I is multi-set additive if and only if it is enhanced additive.

Clearly, additivity implies enhanced additivity. While we can see that the MIV measure family satisfies the additivity and the enhanced additivity, it is not the case for the I_M measure as showed below.

Proposition 1. *I_M is neither additive nor enhanced additive.*

Proof. Consider the counter-example: $K_1 = \{a, \neg a\}$, $K_2 = \{b, \neg b\}$, and $K = K_1 \cup K_2$. It is easy to check that K and K_i $(i = 1, 2)$ satisfy the conditions of additivity and enhanced additivity. But we have $I_M(K_1 \cup K_2) = 3$ while $I_M(K_1) = I_M(K_2) = 1$. $\square$

[1]The Additivity condition is named Decomposability in [15].
[2]We denote a partition $\{A, B\}$ of a set C by $C = A \oplus B$, i.e., $C = A \cup B$ and $A \cap B = \emptyset$.

Indeed, the following theorem states that under certain constraints, $MSSes$ is multiplicative instead of additive.

PROPOSITION 2. *Let K_1 and K_2 be two knowledge bases such that $\{\alpha \in MUSes(K_i)\} \cap \{\beta \in MUSes(K_j)\} = \emptyset$. Then, $|MSSes(K_1 \cup K_2)| = |MSSes(K_1)| \times |MSSes(K_2)|$.*

As the enhanced additivity gives a more intuitive characterization of interaction among sub-sets to be added up, in the following, we are interested in restoring the enhanced additivity of the I_M measure. To simplify terminology, in the rest of the paper, we call the enhanced additivity simply Additivity by default unless other claims are made.

To reach this goal, let us first define three fundamental concepts: *MUS-graph*, *MUS-decomposition*, and *elementary MSS*.

DEFINITION 8 (MUS-GRAPH). *Given an inconsistent knowledge base K, the MUS-graph of K, denoted $\mathcal{G}_{\mathcal{MUS}}(K)$, is an undirected graph where:*

(1) $MUSes(K)$ is the set of vertices, and

(2) $\forall \mathcal{M}, \mathcal{M}' \in MUSes(K)$, $\{\mathcal{M}, \mathcal{M}'\}$ is an edge iff $\mathcal{M} \cap \mathcal{M}' \neq \emptyset$.

A MUS-graph of a knowledge base K gives us a structural representation of its interconnected minimal unsatisfiable subsets.

Moreover, $\mathcal{G}_{\mathcal{MUS}}(K)$ leads to a partition of a knowledge base K, named *MUS-decomposition*, as defined below.

DEFINITION 9 (MUS-DECOMPOSITION). *Let K be a knowledge base, and $\{\mathcal{C}_1, \ldots, \mathcal{C}_p\}$ with $\mathcal{C}_i \subseteq K$ for $1 \leq i \leq p$. $\{\mathcal{C}_1, \ldots, \mathcal{C}_p\}$ is the MUS-decomposition of K iff $\{\mathcal{C}_1, \ldots, \mathcal{C}_p\}$ is the set of the connected components of $\mathcal{G}_{\mathcal{MUS}}(K)$.*

A MUS-decomposition $\{\mathcal{C}_1, \ldots, \mathcal{C}_p\}$ of a knowledge base K represents a partition of the minimal unsatisfiable subsets of K into connected components.

By the fact that $MUSes(K) \neq \emptyset$ and the uniqueness of connected components of a graph, we can easily see:

PROPOSITION 3. *MUS-decomposition exists and is unique for an inconsistency knowledge base.*

Definition 9 allows us to associate to a given knowledge base K a set of sub-bases $K_1, \ldots, K_p$ such that the elements of each sub-base K_i are the formulae of the connected component $\mathcal{C}_i$.

In the following, we present an alternative to the inconsistency measure I_M so as to make it additive. To this end, we introduce the concept of *elementary MSS* by using the MUS-decomposition.

DEFINITION 10. *Let K be a knowledge base, $K' \subseteq K$, and $\{\mathcal{C}_1, \ldots, \mathcal{C}_p\}$ the MUS-decomposition of K. K' is an elementary MSS of K iff there exists a connected component $\mathcal{C}_i$ of K such that $K' \in MSSes(\mathcal{C}_i)$. We denote by $EMSS(K)$ the set of all elementary MSS of K, i.e., $EMSS(K) = \bigcup_{i=1}^{p} MSSes(K_i)$.*

That is, an elementary maximal consistent subset should be locally restricted by a connected component of $MUSes(K)$.

EXAMPLE 2. [3] *Let $K = \{a \wedge d, \neg a, \neg b, b \vee \neg c, \neg c \wedge d, \neg c \vee e, c, \neg e, e \wedge d\}$. The MUS-decomposition of K contains two subsets, $\mathcal{C}_1 = \{\mathcal{M}_1\}$, and $\mathcal{C}_2 = \{\mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_5\}$ where $\mathcal{M}_1 = \{\neg a, a \wedge d\}$, $\mathcal{M}_2 = \{c, \neg b, b \vee \neg c\}$, $\mathcal{M}_3 = \{c, \neg c \wedge d\}$, $\mathcal{M}_4 = \{\neg c \vee e, c, \neg e\}$, and $\mathcal{M}_5 = \{\neg e, e \wedge d\}$ (see figure 1). Then, $EMSS(K) = \{\{a \wedge d\}, \{\neg a\}, \{\neg b, b \vee \neg c, \neg c \wedge d, \neg c \vee e, e \wedge d\}, \{\neg b, b \vee \neg c, \neg c \wedge d, \neg c \vee e, \neg e\}, \{b \vee \neg c, \neg c \vee e, c, e \wedge d\}, \{\neg b, \neg c \vee e, c, e \wedge d\}, \{b \vee \neg c, c, \neg e\}, \{\neg b, c, \neg e\}\}$.*

Now, we use the notion of $EMSS$ to define an alternative of the I_M measure, named I'_M:

$$I'_M(K) = \begin{cases} |EMSS(K)| + |SelfC(K)| & \text{if } K \vdash \bot; \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 4. *the I'_M measure is additive.*

PROOF. The conclusion follows immediately from the fact that both $|EMSS(K)|$ and $|SelfC(K)|$ are additive. □

That is, by taking into account the connections between minimal inconsistent subsets, MUS-decomposition gives us a way to define an inconsistency measure which still satisfies the additivity.

4. MUS-DECOMPOSITION BASED INCONSISTENCY MEASURE

In this section, we use the MUS-decomposition of a knowledge base, defined in the previous section, to estimate the responsibility of each formula to the inconsistency of its base.

Given a MUS-graph of a knowledge base K, a distance, as defined next, is an assignment of a real number to each MUS pair of K.

DEFINITION 11. *Let K be a knowledge base. We denote $d_{MUS}(\mathcal{M}, \mathcal{M}')$ the shortest path between $\mathcal{M}$ and $\mathcal{M}'$ in the MUS-graph $\mathcal{G}_{\mathcal{MUS}}(K)$.*

As easily seen, $d_{MUS}(\mathcal{M}, \mathcal{M}')$ is a distance, that is, it satisfies: (1) $d_{MUS}(\mathcal{M}, \mathcal{M}') \geq 0$; (2) $d_{MUS}(\mathcal{M}, \mathcal{M}') = d_{MUS}(\mathcal{M}', \mathcal{M})$; and (3) $d_{MUS}(\mathcal{M}_1, \mathcal{M}_2) \leq d_{MUS}(\mathcal{M}_1, \mathcal{M}_3) + d_{MUS}(\mathcal{M}_3, \mathcal{M}_2)$. So, we call it the distance between $\mathcal{M}$ and $\mathcal{M}'$.

Next, we will extend Definition 11 to compute the distance between a formula and a MUS.

DEFINITION 12. *Let K be a knowledge base, $\alpha \in K \setminus free(K)$, and $\mathcal{M} \in MUSes(K)$. The distance between α and $\mathcal{M}$ is defined as $d_{MUS}(\alpha, \mathcal{M}) = min\{d_{MUS}(\mathcal{M}, \mathcal{M}') \mid \alpha \in \mathcal{M}'\}$.*

In fact, the distance between a given formula $\alpha \in K$ and a MUS $\mathcal{M}$ corresponds to the shortest path from α to $\mathcal{M}$ along a sequence of intersecting $MUSes$. Note that if α and $\mathcal{M}$ do not belong to the same connected component of $\mathcal{G}_{\mathcal{MUS}}(K)$, this means that $\mathcal{M}$ is not reachable from α and in this case, the distance is assigned an infinite value, i.e., $d_{MUS}(\alpha, \mathcal{M}) = +\infty$.

EXAMPLE 3. *(Example 2 Contd.) Let $\alpha = \neg b$, we have*

$d_{MUS}(\alpha, \mathcal{M}_2) = 0$	$d_{MUS}(\alpha, \mathcal{M}_3) = 1$
$d_{MUS}(\alpha, \mathcal{M}_4) = 1$	$d_{MUS}(\alpha, \mathcal{M}_5) = 2$

[3] For a better display of the new notions introduced in this paper, we use a different example from the illustrative one given in Section 1.

We note that the distance d_{MUS} allows for an ordering over the minimal inconsistent subsets of K according to their distances from α.

In the following, we quantify the inconsistency value of α in the light of the distances from α to reachable *MUSes* of K. Indeed, for each formula belonging to some MUSes, there exists at least one such a finite distance. To compare different formulae by their inconsistency values, only finite distances are meaningful. For free formulae, all the distances will be $+\infty$. But by *Free Formula Independence* principle, they should not be contributors to inconsistency anyway. Let us note $d^{max}_{MUS}(\alpha) = max\{d_{MUS}(\alpha, \mathcal{M}) \mid \mathcal{M} \in MUSes(K),\ d_{MUS}(\alpha, \mathcal{M}) \neq +\infty\}$. Note that the maximum distance is not more than the cardinality of the connected component that α belongs to, that is, $d^{max}_{MUS}(\alpha) < |\mathcal{C}|$.

DEFINITION 13. *Let K be a knowledge base, and $\alpha \in K$. Write $\mathcal{S}_{MUS}(\alpha, k) = \{\mathcal{M} \in MUSes(K) \mid d_{MUS}(\alpha, \mathcal{M}) = k\}$. $\mathcal{SQ}_{MUS}(\alpha)$ is defined as follows: $\mathcal{SQ}_{MUS}(\alpha) = \{\mathcal{S}_{MUS}(\alpha, 0), \mathcal{S}_{MUS}(\alpha, 1), \ldots, \mathcal{S}_{MUS}(\alpha, d^{max}_{MUS}(\alpha))\}$.*

Note that $\mathcal{S}_{MUS}(\alpha, k)$ represents the set of MUSes with a distance k from α, and $\mathcal{SQ}_{MUS}(\alpha)$ gives the sequence of MUSes distributed in terms of the distance d_{MUS}.

EXAMPLE 4. *(Example 2 Contd.) For $\alpha = \neg b$, we have $\mathcal{S}_{MUS}(\alpha, 0) = \{\mathcal{M}_2\}$, $\mathcal{S}_{MUS}(\alpha, 1) = \{\mathcal{M}_3, \mathcal{M}_4\}$, $\mathcal{S}_{MUS}(\alpha, 2) = \{\mathcal{M}_5\}$. Then, $\mathcal{SQ}_{MUS}(\alpha) = \{\{\mathcal{M}_2\}, \{\mathcal{M}_3, \mathcal{M}_4\}, \{\mathcal{M}_5\}\}$. Figure 1 depicts a graphical representation of the connected components of K. Also, Figure 2 represents the distribution of MUSes according to their distances to α. We remark that the MUSes are arranged into different levels.*

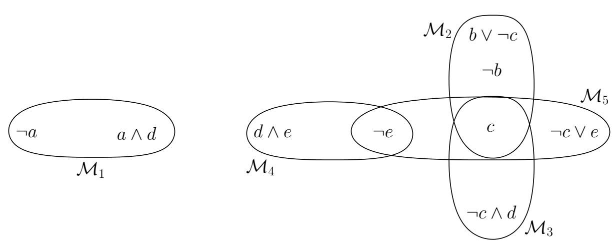

Figure 1: MUS-graph of K

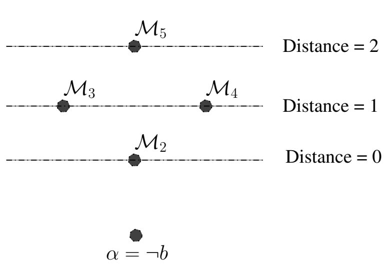

Figure 2: Distribution of MUSes according to α

It is easy to check that the $MIV_{\#}$ measure is just a characterization of the set $\mathcal{S}_{MUS}$ when applying uniquely the minimum distance from the formulae. More formally,

PROPOSITION 5. *Let K be a knowledge base, and $\alpha \in K$. Then, $MIV_{\#}(K, \alpha) = |\mathcal{S}_{MUS}(\alpha, 0)|$.*

Proposition 5 shows that the $MIV_{\#}$ value considers only the nearest neighbors of α. However, this measure is not sufficiently discriminating for our purposes, since it takes into account only the first level of MUSes. Indeed, let us consider the formulae $d \wedge e$, and $\neg c \vee e$ of Example 2. According to $MIV_{\#}$ or MIV_C measure, these two formulae have the same inconsistency value. However, according to Figure 1 $d \wedge e$ and $\neg c \vee e$ do not have the same structural properties. Indeed, the MUS containing $\neg c \vee e$ is more connected than the one of $d \wedge e$. Hence, one has to go beyond the nearest neighbors to obtain a finer-grained measure.

A first inconsistency measure can be defined as follows:

DEFINITION 14. *Let K be a knowledge base, and $\alpha \in K$. The inconsistency value of α is defined as:*

$$DIM_C(\alpha, K) = \frac{|\mathcal{S}_{MUS}(\alpha, 0)|}{d^{max}_{MUS}(\alpha) + 1}$$

Unlike $MIV_{\#}$, the DIM_C value takes into account the structure of connected components by considering the nearest and the farthest MUSes. More precisely, the DIM_C measure aims to assign a higher value to a formula that has numerous nearest neighbors with the remaining *MUSes* concentrated around it. Put differently, while two formulae have the same number of neighbors of distance 0, the distance from the farthest MUSes allows us to find out the most inconsistent one. Note that if α is a self-contradictory formula, the DIM_C measure takes value one, i.e., $DIM_C(\alpha, K) = 1$.

Let us now illustrate the behavior of the DIM_C measure in the next example.

EXAMPLE 5. *(Example 2 Contd.) It is not hard to see:*
$DIM_C(a \wedge d, K) = 1$ $\quad$ $DIM_C(\neg a, K) = 1$
$DIM_C(\neg b, K) = \frac{1}{3}$ $\quad$ $DIM_C(\neg c \wedge d, K) = \frac{1}{3}$
$DIM_C(\neg e, K) = 1$ $\quad$ $DIM_C(b \vee \neg c, K) = \frac{1}{3}$
$DIM_C(d \wedge e, K) = \frac{1}{3}$ $\quad$ $DIM_C(c, K) = \frac{3}{2}$
$DIM_C(\neg c \vee e, K) = \frac{1}{2}$

Notice that now we can make a distinction between $d \wedge e$, and $\neg c \vee e$, since $DIM_C(d \wedge e, K) < DIM_C(\neg c \vee e, K)$.

However, the problem remains between the formulae $d \wedge e$, and $\neg b$. In order to make the measure more accurate, we propose to extend our framework by not only considering the farthest MUSes, but the whole structure of the connected components, which leads to a more useful inconsistency measure defined as follows:

DEFINITION 15. *Let K be a knowledge base, and $\alpha \in K$. The inconsistency value of α is defined as:*

$$DIM_H(\alpha, K) = \sum_{\substack{\mathcal{M} \in MUSes(K) \\ d_{MUS(\alpha, \mathcal{M}) \neq +\infty}}} \frac{1}{d_{MUS}(\alpha, \mathcal{M}) + 1}$$

We can see more clearly by the following example that DIM_H gives a more precise view of the conflict brought by each formula.

EXAMPLE 6. *(Example 2 Contd.) We have:*
$DIM_H(a \wedge d, K) = 1$ $\quad$ $DIM_H(\neg a, K) = 1$
$DIM_H(\neg b, K) = \frac{7}{3}$ $\quad$ $DIM_H(\neg c \wedge d, K) = \frac{7}{3}$
$DIM_H(\neg e, K) = 3$ $\quad$ $DIM_H(b \vee \neg c, K) = \frac{7}{3}$
$DIM_H(d \wedge e, K) = \frac{13}{6}$ $\quad$ $DIM_H(c, K) = \frac{7}{2}$
$DIM_H(\neg c \vee e, K) = \frac{7}{3}$

Using the DIM_H measure, the formula $d \wedge e$ has now an inconsistency value $\frac{13}{6}$ less than that of $\neg b$ which is $\frac{7}{3}$.

The DIM_H measure could be refined by using the following notion of a weighting function that assigns a weight to each MUS in the MUS-graph. The idea is that a weight represents the significance of a MUS with respect to their distance from a given formula, thus leading to a better assignment of inconsistency responsibility to formulae. These weights can take into consideration other criteria like the degree of each MUS in the MUS-graph. A general definition is stated as follows.

Definition 16. *Let K be a knowledge base, and $w(\mathcal{M}) \in \mathbb{R}$ a given weight function. The inconsistency value of α is defined as:*

$$DIM_W(\alpha, K) = \sum_{\substack{\mathcal{M} \in MUSes(K) \\ d_{MUS(\alpha,\mathcal{M}) \neq +\infty}}} \frac{w(\mathcal{M})}{d_{MUS}(\alpha, \mathcal{M}) + 1}$$

The following result shows that the DIM_W measure can be expressed by using the sequence $\mathcal{SQ}_{MUS}(\alpha)$. The idea is that $w(\mathcal{M})$ only depends on the distance between $\mathcal{M}$, and α.

Proposition 6. *Let K be a knowledge base, and $\alpha \in K$. Then, the following equation holds:*

$$DIM_W(\alpha, K) = \sum_{\mathcal{S}_{MUS} \in \mathcal{SQ}_{MUS}} w(i) \times \frac{|\mathcal{S}_{MUS}(\alpha, i)|}{(i+1)}$$

Note that $w(i)$ represents the weight associated to each MUS at distance i from α.

4.1 Measuring Inconsistency of Sub-bases

Instances of a DIM (such as those given in Definitions 14, 15, and 16) can be obviously extended to a set of formulae. For this purpose it will be convenient to define the distance between a subset $K' \subseteq K$, and a MUS $\mathcal{M} \in MUSes(K)$.

Definition 17. *Let K be a knowledge base, K' a subset of K, and $\mathcal{M} \in MUSes(K)$. The distance between K', and $\mathcal{M}$ is defined as:*

$$d_{MUS}(K', \mathcal{M}) = min\{d_{MUS}(\alpha, \mathcal{M}) \mid \alpha \in K'\}$$

Clearly, the set of $MUSes$ can be partitioned into different subsets according to their distances to the subset K'. Here, we denote by $d^{max}_{MUS}(K')$ the maximum distance of $MUSes$ from K'.

Now, the inconsistency measure of K' through DIM_C value is defined as follows:

Definition 18. *Let K be a knowledge base, and $K' \subseteq K$. The inconsistency degree of K' is defined as:*

$$DIM_C(K', K) = \frac{|\mathcal{S}_{MUS}(K', 0)|}{d^{max}_{MUS}(K') + 1}$$

Interestingly, we note that while considering more than one formula, different connected components should be taken into account. In the light of the property of additivity, $DIM_C(K', K)$ can be rewritten as stated in Proposition 7.

Proposition 7. *Let K be a knowledge base, $K' \subseteq K$, and $\{\mathcal{C}_1, \ldots, \mathcal{C}_p\}$ the MUS-decomposition of K. Then,*

$$DIM_C(K', K) = \sum_{i=1}^{i=p} DIM_C(K' \cap K_i, K)$$

Example 7. *Let us consider again the knowledge base $K = \{a \wedge d, \neg a, \neg b, b \vee \neg c, \neg c \wedge d, \neg c \vee e, c, \neg e, e \wedge d\}$. Then, $DIM_C(\{a \wedge d, c\}, K) = 1 + \frac{3}{2} = \frac{5}{2}$.*

Similarly, DIM_C, DIM_H, and DIM_W inconsistency values can be naturally extended to a set of formulae. In particular, if we consider the case $K' = K$, then $DIM_C(K) = \frac{|\mathcal{S}_{MUS}(K,0)|}{d^{max}_{MUS}(K)+1} = |MUSes(K)|$, which corresponds to the I_{MI} measure. The same result is obtained when using DIM_H, i.e., $DIM_H(K) = |MUSes(K)|$.

4.2 Monotonicity

As seen earlier, the DIM_C value combines the distances to the nearest $MUSes$, and the inverse of that to the farthest in order to quantify the participation of each formula in the inconsistencies. Note that adding new formulae to a knowledge base may grow the distance to the farthest MUS, and consequently the DIM_C value decreases. This means that DIM_C is not monotonic. For illustration, consider the knowledge base $K = \{a, \neg a \wedge b, \neg b \wedge c, \neg c \wedge d, \neg d \wedge e\}$. Then, we have $DIM_C(a) = \frac{1}{4}$. Now if we add to K the formula $\neg e$, then the value of a in $K \cup \{\neg e\}$ becomes $\frac{1}{5}$.

In contrast, DIM_H counts the inverse of all distances from a formula to each MUS. Moreover, adding new formulae cannot decrease the number of $MUSes$, and cannot increase the distance of existing $MUSes$ from the formula, thus the inverse of the distances will be non-dereasing. Consequently, the DIM_H measure is monotonic.

5. MEASURING INCONSISTENCIES OF A WHOLE BASE

This section is devoted to the definition of an inconsistency measurement for a whole knowledge base.

To address this question, let us firstly give a general characterization of our measure with respect to the additivity property. Then, we discuss different measures obtained by different instantiations of the general case.

Definition 19. *Let K be a knowledge base, and $\mathcal{CC} = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ the connected components of K. The inconsistency measure of K, denoted $I_{\mathcal{CC}}(K)$, is defined as $I_{\mathcal{CC}}(K) = \sum_{i=1}^{n} \delta(\mathcal{C}_i)$ where δ is a function valuing in $\mathbb{R}$.*

The general definition given above allows for a range of possible measures. Next we first instantiate some $I_{\mathcal{CC}}$ measures by varying the δ function. The simplest one is obtained when $\delta(\mathcal{C}_i) = 1$. In this case, we get a measure that assigns to K the number of its connected components, i.e., $I_{\mathcal{CC}}(K) = |\mathcal{CC}|$. Note that this measure is not monotonic. Indeed, adding new formulae to the knowledge base can decrease the number of connected components of the base. For instance, let us consider the knowledge base $K = \{a, \neg a, b, \neg b\}$ that contains two connected components $\mathcal{C}_1 = \{a, \neg a\}$, and $\mathcal{C}_2 = \{b, \neg b\}$; now adding the formula $a \vee b$ to K leads to a new knowledge base containing a unique connected component $\mathcal{C} = \{a, \neg a, b, \neg b, a \vee b\}$. Note that this simple measure considers each connected component as an inseparable entity.

Moreover, when we consider $\delta(\mathcal{C}_i) = |\mathcal{C}_i|$, the $I_{\mathcal{CC}}$ measure leads to an existing inconsistency measure. More precisely, as $|\mathcal{C}_i|$ corresponds exactly to the number of $MUSes$ involved in the connected component $\mathcal{C}_i$, it is obvious to see that $I_{\mathcal{CC}}(K)$ is equal to I_{MI} measure, i.e., $I_{\mathcal{CC}}(K) = |MUSes(K)|$. This second value is of little interest, since it states exactly the fact that the inconsistency value only takes into account the number of the minimal inconsistent subsets of a base.

In the following, we propose to deeply explore the properties of additivity, and monotony to define a new interesting inconsistency measure.

Definition 20. *Let K be a knowledge base, and $K_1, \ldots, K_n$ subsets of K. The set $\{K_1, \ldots, K_n\}$ is called a conditional independent MUS partition of K if the following conditions are satisfied:*

(1) $K_i \vdash \bot$, for $1 \leq i \leq n$,

(2) $MUSes(K_1 \cup \ldots \cup K_n) = \bigoplus_{1 \leq i \leq n} MUSes(K_i)$,

(3) $K_i \cap K_j = \emptyset$, $\forall i \neq j$.

According to Definition 20, K can be written as $K = K_1 \cup \ldots \cup K_n \cup R$ where R is a subset of K, and $\{K_1, \ldots, K_n\}$ is a conditional independent MUS partition of K. That is, when removing the set of formulae R from K the remaining base can be partitioned into sub-bases $K_1, \ldots, K_n$ satisfying conditions (1), (2), and (3) (Definition 20).

In general, for a given knowledge base K, there exist different subsets $R \subseteq K$ such that Definition 20 holds. Moreover, if $K = K_1 \cup \ldots \cup K_n \cup R$, then there exists $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\} \subseteq MUSes(K)$ being a conditional independent MUS partition of K. In other words, $K = \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_n \cup R'$ where $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$, $\forall\ 1 \leq i \neq j \leq n$, and $MUSes(\mathcal{M}_1 \cup \ldots \cup \mathcal{M}_n) = \oplus_{1 \leq i \leq n} \mathcal{M}_i$. Indeed, it is sufficient to pick a MUS $\mathcal{M}_i$ from each K_i since $K_i \vdash \bot$, and consider $R' = R \cup \{K_1 \setminus \mathcal{M}_1\} \cup \ldots \cup \{K_n \setminus \mathcal{M}_n\}$.

Let us now characterize our inconsistency measure I in the light of both additivity, and monotony properties. Using Definition 20, $I(K) = I(K_1 \cup \ldots \cup K_n \cup R)$. Using monotony property, we have $I(K) \geq I(K_1 \cup \ldots \cup K_n)$. Finally by additivity, we conclude that $I(K) \geq I(K_1) + \ldots + I(K_n)$.

Now, let us denote by $\mu_{max}(K)$ the maximal cardinality of sets $\{K_1, \ldots, K_n\}$ satisfying the conditions (1), (2), and (3) of Definition 20. That is, $\mu_{max}(K)$ corresponds to the number of maximal connected components that can be obtained while removing some formulae from K, i.e., the maximum value taken by n. By considering the maximal conditional independent partition value μ_{max}, one can deduce that if the measure I is additive then, $I(K) \geq I(K_1) + \ldots + I(K_{\mu_{max}(K)})$. Now, by using this bound one can define a new inconsistency measure as stated in Definition 21.

Definition 21. *Let K be a knowledge base. We define the inconsistency measure of K as:*

$$I_{\mathcal{CC}}(K) = \mu_{max}(K)$$

So now we can show that:

Proposition 8. *The inconsistency measure $I_{\mathcal{CC}}$ is additive, and monotonic.*

Proof. Let K, and K' be two knowledge bases such that $K = K_1 \cup \ldots \cup K_{\mu_{max}(K)} \cup R$. Then, $K \cup K' = K_1 \cup \ldots \cup K_{\mu_{max}(K)} \cup R \cup K'$. Hence, $K \cup K' = K_1' \cup \ldots \cup K_{\mu_{max}(K)}' \cup R'$. Thus, $\mu_{max}(K \cup K') \geq \mu_{max}(K)$ which proves that $I_{\mathcal{CC}}$ is monotonic. Suppose now that $(K \setminus free(K)) \cap (K' \setminus free(K')) = \emptyset$, and $MUSes(K) \cap MUSes(K') = \emptyset$, it is easy to conclude that $\mu_{max}(K \cup K') = \mu_{max}(K) + \mu_{max}(K')$. As consequently, $I_{\mathcal{CC}}$ is additive. □

Example 8. *Let us consider $K = \{a, \neg a, a \vee b, \neg b, b, c, \neg c \wedge d, \neg d \wedge e \wedge f, \neg e, \neg f\}$. The figure 3 depicts its connected components. K contains two connected components $\mathcal{C}_1$, and $\mathcal{C}_2$ such that $\mathcal{C}_1 = \{a, \neg a, a \vee b, \neg b, b\}$, and $\mathcal{C}_2 = \{c, \neg c \wedge d, \neg d \wedge e \wedge f, \neg e, \neg f\}$. The conditional independent MUS partition of $\mathcal{C}_1$ of maximum size is equal to 2, and can be obtained from $\mathcal{C}_1$ by removing $a \vee b$. For $\mathcal{C}_2$, for all removed subsets of $\mathcal{C}_2$, the number of resulting connected components can not exceed 1. Then, we have $I_{\mathcal{CC}}(K) = I_{\mathcal{CC}}(\mathcal{C}_1) + I_{\mathcal{CC}}(\mathcal{C}_2) = 3$.*

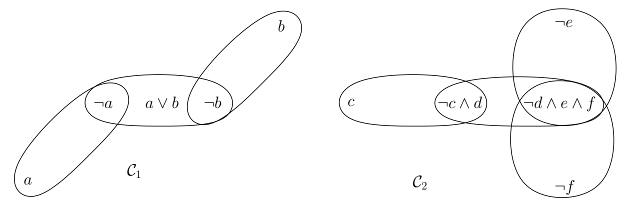

Figure 3: Connected components of the knowledge base K

Let us note that among measures that are additive, monotonic, and MinInc, I_{CC} is the smallest one, as shown below.

Proposition 9. *For any I that is additive, monotonic, and MinInc, $I_{CC}(K) \leq I(K)$.*

Proof. Indeed, an additive, and monotonic measure must attribute to a knowledge base a value greater than $I(K_1) + \ldots + I(K_{\mu_{max}(K)})$. When a measure satisfies MinInc, and monotonicity, $I(K_i) \geq 1$. Hence, $I(K) \geq \mu_{max}(K)$. □

Moreover, Proposition 10 shows the relationship between the maximal conditional independent MUS partition, and the smallest minimal hitting set.

Proposition 10. *Let K be a knowledge base. Then,*

$$I_{CC}(K) \leq LB_{HS}(K)$$

Proof. As K can be partitioned into $\mu_{max}(K)$ independent components by removing some formulae, then a minimal hitting set of K must contain at least one formula from each independent component of the maximal partition. Hence, $\mu_{max}(K) \leq LB_{HS}(K)$. □

Example 9. *Let us consider again Example 8, and its second connected component $\mathcal{C}_2$. We have $LB_{HS}(K) = 2$ while the conditional independent MUS partition of K can not exceed 1.*

6. CONCLUSION

We proposed in this paper a new framework for defining inconsistency values that allow to associate each formula with its degree of responsibility for the inconsistency of a whole knowledge base. This approach is based on the correlation between minimal inconsistent subsets which is shown a useful way to quantify the amount of inconsistencies in a finer way. We showed that such a framework can be extended to quantify the inconsistency of a whole knowledge base. We also proposed an enhanced additivity property to better capture its intuition according to the debat existing in the literature.

In the future, we plan to investigate deeply the architecture of the connected components. For instance, we plan to use logical argumentation theory [4] to deepen the analysis of the graph representation of a knowledge base [3]. We are also going to investigate results coming from graph theory in order to offer a finer grained evaluation of the inconsistencies. The theoretical complexity of our inconsistency measures, and practical algorithms are under investigation.

7. ACKNOWLEDGMENTS

The first author benefits from the support of both CNRS and OSEO within the ISI project Pajero.

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