Compiling Constraint Networks into Multivalued Decomposable Decision Graphs

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Abstract

We present and evaluate a top-down algorithm for compiling finite-domain constraint networks (CNs) into the language MDDG of multivalued decomposable decision graphs. Though it includes Decision-DNNF as a proper subset, MDDG offers the same key tractable queries and transformations as Decision-DNNF, which makes it useful for many applications. Intensive experiments showed that our compiler cn2mddg succeeds in compiling CNs which are out of the reach of standard approaches based on a translation of the input network to CNF, followed by a compilation to Decision-DNNF. Furthermore, the sizes of the resulting compiled representations turn out to be much smaller (sometimes by several orders of magnitude).

1 Introduction

Constraint Programming (CP) has long been recognized as a paradigm of choice for representing and solving combinatorial problems [Rossi *et al.*, 2006]. Knowledge about the problem is represented in a compact and intuitive way, using a *constraint network* (CN), which involves a set of variables associated with their domain of values, and a collection of constraints specifying that some subsets of values cannot be used together. Despite its undoubtable success in IA, one of the key remaining challenges of CP is to provide *performance guarantees* for query answering, which often amounts to solving instances of NP-hard problems. As emphasized in [Freuder and O'Sullivan, 2014], this issue is critical in *on-line* applications, such as configuration softwares [Junker, 2006] and recommender systems [Cambazard *et al.*, 2010], where queries supplied "on the fly" by users, are to be answered in real time.

The aim of this paper is to address this challenge using *knowledge compilation* [Darwiche and Marquis, 2002]. The overall idea is to convert a constraint language into a target compilation language that supports inference tasks (often classified as queries and transformations) in polynomial time. Thus, while many queries are intractable when the input is a CN, they become tractable from a compiled representation of it, enabling on-line performance guarantees when the compiled representation remains small enough.

Specifically, we present a top-down algorithm cn2mddg for compiling finite-domain CNs into multivalued decomposable decision graphs. The input of cn2mddg is a CN represented in the XCSP 2.1 format [Roussel and Lecoutre, 2009]. The output of our compilation algorithm is a representation of the solutions of the CN in the language MDDG of multivalued decomposable decision graphs. MDDG is precisely the extension to non-Boolean domains of the language DDG [Fargier and Marquis, 2006] also known as Decision-DNNF [Oztok and Darwiche, 2014]: it is based on decomposable $\wedge$-nodes and (multivalued) decision nodes. Similarly to Decision-DNNF, the MDDG language offers a number of tractable queries, including (possibly weighted) solution finding and counting, solution enumeration (solutions can be enumerated with polynomial delay), and optimization w.r.t. a linear objective function. It also offers tractable transformations, especially the conditioning one (i.e., the instantiation of variables, and more generally, the addition of unary constraints).

cn2mddg benefits from a specific caching technique, a new variable ordering heuristic based on betweenness centrality and it detects universal constraints during the search in order to perform additional simplifications. We performed an intensive evaluation of cn2mddg on a number of benchmarks (173) from several data sets (15). Given the availability of Decision-DNNF compilers, a way to compile CNs is to follow a translate-then-compile schema: one first encodes the input network into a CNF formula, then one takes advantage of a compiler like c2d [Darwiche, 2004] or Dsharp [Muise *et al.*, 2012] to turn the resulting formula into a Decision-DNNF representation. Based on the results reported from our experimentations, it turns out that whatever the encodings used, both the huge number of Boolean variables in the generated CNF formulae, and the structure loss inherent to the CNF format (compared to the constraint network one) make the translate-then-compile approaches impractical in many cases. Contrastingly, our compiler cn2mddg proved much more robust since it succeeded in compiling many CNs which are out of reach of the translate-then-compile approaches. Moreover, the sizes of the resulting compiled representations turn out to be much smaller (sometimes by several orders of magnitude).

The run-time code of our compiler, as well as the translators and the benchmarks used in our experiments, and additional empirical results, can be downloaded from www.cril.fr/KC/

2 Formal Preliminaries

A *finite-domain constraint network (CN)* is a triple $\mathcal{N} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$ consisting of a set $\mathcal{X} = \{X_1, \cdots, X_n\}$ of *variables*, a set $\mathcal{D} = \{D_1, \cdots, D_n\}$ of *domains*, and a set $\mathcal{C} = \{C_1, \cdots, C_m\}$ of *constraints*. Each domain D_i is a finite set containing the possible values of X_i. Each constraint C_j characterizes the combinations of values satisfying it. Formally, $C_j = (S_j, R_j)$, where $S_j = \{X_{j_1}, \cdots, X_{j_k}\}$ is a subset of variables from $\mathcal{X}$, called the *scope* of C_j, and R_j is a predicate over the Cartesian product $D_{j_1} \times \cdots \times D_{j_k}$, called the *relation* of C_j. R_j can be represented extensionally by the list of its satisfying tuples (or dually, by the list of its forbidden tuples), or intensionally by an oracle, i.e., a mapping from $D_{j_1} \times \cdots \times D_{j_k}$ to $\{0, 1\}$ which is supposed to be computable in time polynomial in its input size. The *arity* of a constraint is given by the size of its scope. Constraints of arity 2 are called *binary* and constraints of arity greater than 2 are called *non-binary*.

Example 1 *Let $\mathcal{N}$ be the CN given by four variables X_1, X_2, X_3, and X_4, each of them being defined on the same domain $\{0, 1, 2\}$, and three constraints C_1, C_2, and C_3, specified by the following mathematical statements:*

- $C_1 = (X_1 \neq X_2)$*;*
- $C_2 = (X_2 = 0) \vee (X_2 = 1) \vee (X_2 = X_3 + X_4 + 1)$*;*
- $C_3 = (X_3 > X_4)$.

Given a subset S of variables from $\mathcal{X}$, a *(decision) state* $\boldsymbol{s}$ over S is a mapping that associates with each variable X_i in S a subset $\boldsymbol{s}(X_i)$ of values in D_i. In what follows, states are often noted as union of elementary assignments, i.e., sets of the form $\{\langle X_i, x_j \rangle\}$, where $x_j \in \boldsymbol{s}(X_i)$. $\mathit{scope}(\boldsymbol{s})$ denotes the set S of variables over which $\boldsymbol{s}$ is defined. A state $\boldsymbol{s}$ is *partial* if $\mathit{scope}(\boldsymbol{s})$ is a proper subset of $\mathcal{X}$; otherwise, $\boldsymbol{s}$ is called a *full* state. A variable X_i in $\mathit{scope}(\boldsymbol{s})$ is *instantiated* if $\boldsymbol{s}(X_i)$ is a singleton set. The set of instantiated variables in $\boldsymbol{s}$ is noted $\mathit{single}(\boldsymbol{s})$. As usual, a state $\boldsymbol{s}$ is called an *instantiation* when all its variables are instantiated, i.e., $\mathit{scope}(\boldsymbol{s}) = \mathit{single}(\boldsymbol{s})$.

For a state $\boldsymbol{s}$ and a set of variables $T \subseteq \mathit{scope}(\boldsymbol{s})$, $\boldsymbol{s}[T]$ denotes the *restriction* of $\boldsymbol{s}$ to T, i.e., $\boldsymbol{s}[T]$ is the set $\{\langle X_i, x_j \rangle \in \boldsymbol{s} \mid X_i \in T\}$. An instantiation $\boldsymbol{s}$ *satisfies* a contraint $C_j = (S_j, R_j)$ if $S_j \subseteq \mathit{scope}(\boldsymbol{s})$ and $R_j(x_{j_1}, \ldots, x_{j_k}) = 1$, where $\forall l \in 1, \ldots, k$, $\langle X_{j_l}, x_{j_l} \rangle \in \boldsymbol{s}[S_j]$. A *solution* of a CN $\mathcal{N} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$ is a full instantiation $\boldsymbol{s}$ satisfying all constraints C_j in $\mathcal{C}$. For example, $\boldsymbol{s} = \{\langle X_1, 1 \rangle, \langle X_2, 0 \rangle, \langle X_3, 1 \rangle, \langle X_4, 0 \rangle\}$ is a solution of the CN given at Example 1.

Given a CN $\mathcal{N} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$ and a state $\boldsymbol{s}$ over a subset of $\mathcal{X}$, the *conditioning* $\mathcal{N} \mid \boldsymbol{s}$ of $\mathcal{N}$ by $\boldsymbol{s}$ is the CN $(\mathcal{X}', \mathcal{D}', \mathcal{C}')$ defined as follows: $\mathcal{X}' = \mathcal{X} \setminus \mathit{single}(\boldsymbol{s})$; with each domain D_i in $\mathcal{D}$, one associates the domain $D'_i \in \mathcal{D}'$, where $D'_i = D_i$ if $X_i \notin \mathit{scope}(\boldsymbol{s})$ and $D'_i = \boldsymbol{s}(D_i)$ otherwise; finally, with each constraint $C_j = (S_j, R_j)$ in $\mathcal{C}$, one associates the constraint $C'_j = (S'_j, R'_j)$ in $\mathcal{C}'$, where $S'_j = S_j \setminus \mathit{single}(\boldsymbol{s})$ and R'_j is the restriction of R_j to S'_j.

The primal graph of a CN $\mathcal{N} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$ is the undirected graph G with vertex set $\mathcal{X}$ and edge set $\mathcal{E}$, such that $\{X_p, X_q\} \in \mathcal{E}$ if and only if $\{X_p, X_q\}$ is a subset of the scope S_j of some constraint C_j in $\mathcal{C}$. For instance, the primal graph of the CN given at Example 1 is depicted on Figure 1.

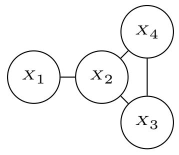


Figure 1: The primal graph of the CN given at Example 1.

3 The MDDG Language

Let us first consider the language MDG of multivalued decision graphs:

Definition 1 (MDG) *Given a finite set $\mathcal{X}$ of finite-domain variables, the (read-once)* MDG *language over $\mathcal{X}$ is the set of all single-rooted directed acyclic graphs Δ, where leaf nodes are labelled by $\top$ (true) or $\bot$ (false), and every internal node is either a $\wedge$-node $N = \wedge(N_1, \ldots, N_i)$ or a decision node N associated with variable $X_i \in \mathcal{X}$, i.e., a deterministic $\vee$-node $N = \vee(N_1, \ldots, N_j)$ such that $D_i = \{x_{i_1}, \ldots, x_{i_j}\}$ and the arc from N to N_k ($k \in 1, \ldots, j$) is labelled by the elementary assignment $\{\langle X_i, x_{i_k} \rangle\}$. The paths of Δ must satisfy the* read-once *property: for every path from the root of Δ to a $\top$ leaf node, and for any variable $X_i \in \mathcal{X}$, no more than one arc can be labelled by an elementary assignment over X_i.*

For every node N in an MDG representation Δ, $\mathit{Var}(N)$ is defined inductively as follows:

- if N is a leaf node, then $\mathit{Var}(N) = \emptyset$;
- if N is a $\wedge$-node $N = \wedge(N_1, \ldots, N_i)$, then $\mathit{Var}(N) = \bigcup_{k=1}^{i} \mathit{Var}(N_i)$;
- if N is a decision node $N = \vee(N_1, \ldots, N_j)$ associated with variable X, then $\mathit{Var}(N) = \{X\} \cup \bigcup_{k=1}^{j} \mathit{Var}(N_k)$.

Let s be a full instantiation over $\mathcal{X}$ and let Δ be a MDG representation over $\mathcal{X}$, rooted at node N. Let $\mathit{eval}(N, \mathrm{s})$ be the MDG representation without any decision node, defined inductively by:

- if N is a leaf node, then $\mathit{eval}(N, \mathrm{s}) = N$;
- if N is a $\wedge$-node $N = \wedge(N_1, \ldots, N_i)$, then $\mathit{eval}(N, \mathrm{s}) = \wedge(\mathit{eval}(N_1, \mathrm{s}), \ldots, \mathit{eval}(N_i, \mathrm{s}))$;
- if N is a decision node $N = \vee(N_1, \ldots, N_j)$ associated with variable X_i, then $\mathit{eval}(N, \mathrm{s}) = \mathit{eval}(N_k, \mathrm{s})$, where $\langle X_i, x_{i_k} \rangle \in \mathrm{s}$.

s is a *solution* of Δ if and only $\mathit{eval}(N, \mathrm{s})$ evaluates to true.

The language MDDG we are interested in is the subset of MDG consisting of *decomposable* representations, those where the children of any $\wedge$-node do not share any variable:

Definition 2 (MDDG) *Given a finite set $\mathcal{X}$ of finite-domain variables, the* MDDG *language over $\mathcal{X}$ is the subset of* MDG *representations Δ, where each $\wedge$-node $N = \wedge(N_1, \ldots, N_i)$ is decomposable, i.e., $\forall k, l \in 1, \ldots, i$, if $k \neq l$, then $\mathit{Var}(N_k) \cap \mathit{Var}(N_l) = \emptyset$.*

The MDDG representation reported on Figure 2 is equivalent to the CN given at Example 1, i.e., they have the same solution set. Nodes with a single child are shunted, and their labels (in the case of decision nodes) are gathered with those of their incoming arcs. The corresponding elementary assignments typically result from propagation (the $\vee$-nodes are not created in that case, we mention them in the definition of MDDG for the ease of exposure). The arcs leading to $\perp$ are not depicted. The $\top$ leaf is duplicated for readability reasons.

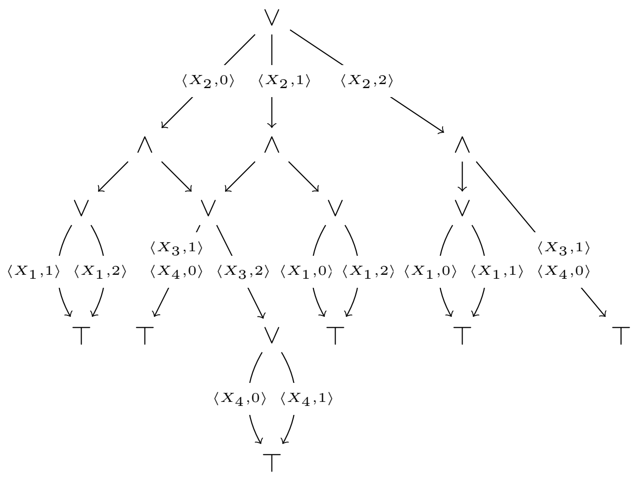


Figure 2: An MDDG equivalent to the CN given at Example 1.

Decision-DNNF [Oztok and Darwiche, 2014; Fargier and Marquis, 2006] corresponds to the proper subset of MDDG where each variable has a Boolean domain. Despite the increase of generality obtained by accepting non-Boolean domains, the key tractable queries and transformations (weighted solution counting, solution enumeration, optimization w.r.t. a linear objective function and conditioning) offered by Decision-DNNF are also offered by MDDG (the polynomial-time (or polynomial-delay) algorithms used to achieve those queries and transformations in the Decision-DNNF case can be extended in a trivial way to the MDDG case).

MDDG is also close to the MDD language considered in [Amilhastre *et al.*, 2014], and to the AOMDD language considered in [Mateescu *et al.*, 2008], but it does not coincide with any of them. Thus, MDD and MDDG are not comparable w.r.t. set-inclusion; on the one hand, MDD consists of non-deterministic structures (more than one outgoing arc of a decision node labelled by a variable X_i can be labelled by the same value from the domain D_i of X_i), while the decision nodes in an MDDG representation are always deterministic ones; on the other hand, MDDG representations include $\wedge$-nodes, while the internal nodes of any MDD representation are decision nodes. Similarly, AOMDD and MDDG are not comparable w.r.t. set-inclusion; on the one hand, AOMDD is suited to the compilation of graphical models (or weighted constraint networks), and as such, it enables the representation of functions the co-domain of which is not Boolean in essence (like utility or cost functions, probability distributions, etc.) while MDDG cannot do it; on the other hand, AOMDD representations are ordered structures (they respect a pseudo-tree induced by a preset elimination order over the variables – this is a crucial requirement for canonicity) while in MDDG there is no such a requirement (MDDG representations are not canonical ones).

4 A Top-Down MDDG Compiler

We developed a top-down compiler cn2mddg which takes as input a CN represented in the XCSP 2.1 format [Roussel and Lecoutre, 2009], and generates a MDDG representation equivalent to it, i.e., having the same solutions. All the basic features offered by the XCSP 2.1 format are taken into account; especially, the constraints can be represented in intension (as "predicates") or in extension (as "relations"); however, only three global constraints are supported by our compiler in its current turn; namely *allDifferent*, *weightedSum* (i.e., linear constraints), and *element*.[1]

The architecture of our cn2mddg compiler is somehow "standard", i.e., close to the one of a top-down compiler suited to Boolean domains, like the c2d compiler (reasoning.cs.ucla.edu/c2d/), or the Dsharp compiler (www.haz.ca/research/dsharp/), both targeting the Decision-DNNF language. Especially, our compiler is search-based: it follows the trace of a search engine [Huang and Darwiche, 2007]. It covers similar techniques as those used in c2d and in Dsharp, including conflict analysis for guiding the search, constraint propagation for simplification purpose, component caching in order to avoid the duplication of identical subparts of the compiled representation, and a dynamic variable ordering heuristic (as in Dsharp which takes advantage of the *vsads* variable selection heuristic).[2] Both the caching technique and the variable ordering heuristic used in cn2mddg are specific to the nature of the input (a CN), which exhibits much more structure than "flat" CNF formulae. Furthermore, our algorithm exploits a specific method for handling universal constraints, enabling additional simplifications to be performed.

Universal constraints. Universal constraints are constraints which are necessarily satisfied whatever the values (in the current domains of the variables) given to the variables of their scopes. Thus, for the CN given at Example 1, contraint C_2 when conditioned by any of the elementary assignments $\{\langle X_2, 0\rangle\}$ or $\{\langle X_2, 1\rangle\}$ becomes universal. At every step of the compilation (i.e., whatever the current decision state), universal constraints are detected. Every constraint $C_j = (S_j, R_j) \in \mathcal{C}$ for which there exists $X_i \in S_j$ such that D_i has been reduced by propagation after the last elementary assignment, is checked for universality. One looks for an instantiation s of the variables of the current scope of C_j to values in their current domains such that s violates R_j; C_j is valid iff one cannot find such an instantiation s. For efficiency reasons, s is searched in a lazy way: when found, s is stored and the next time C_j is checked for universality, s is considered in priority. Once detected, a universal constraint C_j is simply deleted from the current network; obviously, this simplifies the forthcoming treatments (no need to take those constraints into account), favors decomposability (due to the

[1]These constraints can be encoded as "predicates" as well, but then one cannot take advantage of their dedicated propagator.

[2]Contrastingly, in c2d the variable ordering is static.

edge deletion it leads to on the primal graph of the CN), and impacts the variable ordering heuristic. Note that in Decision-DNNF compilers the handling of universal constraints simply amounts to ignoring every clause sharing a literal with the current partial interpretation.

Caching. Caching is a key technique of any compiler computing DAG-based representations. It aims at refraining from solving the same subproblem twice or more, and duplicating parts of the compiled representation. Indeed, due to the (conditional) interchangeability of values in many networks, it is often the case that two distinct decision states $\mathtt{s}_1$ and $\mathtt{s}_2$ considered successively during the search give rise to the same problem, i.e., $\mathcal{N} \mid \mathtt{s}_1$ and $\mathcal{N} \mid \mathtt{s}_2$ are equivalent. In such a case, instead of compiling both networks, it can prove much better to compile $\mathcal{N} \mid \mathtt{s}_1$ only, then to store in a cache an entry corresponding to $\mathcal{N} \mid \mathtt{s}_1$ associated with the root node N of its MDDG representation, and to detect that $\mathcal{N} \mid \mathtt{s}_2$ is equivalent to $\mathcal{N} \mid \mathtt{s}_1$ by looking at each step into the cache: in this case, instead of performing the computationally demanding compilation of $\mathcal{N} \mid \mathtt{s}_2$, it is enough to create an arc pointing to N to do the job. Thus, for the CN $\mathcal{N}$ given at Example 1, the component about $\{X_3, X_4\}$ obtained by dynamic decomposition is the same one for the states $\{\langle X_2, 0\rangle\}$ and $\{\langle X_2, 1\rangle\}$, so there is no need to duplicate it.

However, testing the equivalence of two CNs under states is computationally hard and an exponential number of subproblems have to be considered in general. For these reasons, it is not possible to perform brute-force caching where all non-equivalent $\mathcal{N} \mid \mathtt{s}$ networks encountered during the search would be considered (this would require unmanageable compilation times). Thus, $\mathcal{N} \mid \mathtt{s}_1$ and $\mathcal{N} \mid \mathtt{s}_2$ are detected as "equivalent" when they are identical.

A main issue to be addressed for an efficient caching in practice concerns the size of the entries; preferably, one must keep them as small as possible. In our cache, one first stores the current domains of the current variables, i.e., $\mathtt{s}$ restricted to its unassigned variables. Storing all the current constraints would be too space demanding. Fortunately, this is useless in general. Indeed, every constraint $C_j = (S_j, R_j)$ such that $S_j \cap \mathit{single}(\mathtt{s}) = \emptyset$ does not need to be saved (provided that the initial constraint C_j is available). Furthermore, no constraint $C_j = (S_j, R_j)$ which is binary in the input network needs to be saved, provided that the current network is arc consistent: if no variable from $S_j = \{X_i, X_k\}$ has been instantiated, then the previous case is recovered; if both variables X_i, X_k from S_j have been instantiated, then either C_j is universal or C_j is inconsistent, and there is no need to store it whatever the case; finally if only one variable X_i from S_j has been instantiated (say, to value x_i), then the projection on the remaining (uninstantiated) variable X_k of the restriction of R_j for which $X_i = x_i$ coincides with the restriction of $\mathtt{s}$ to $\{X_k\}$ when the current network is arc consistent.[3] Similarly, there is no need to store the *allDifferent* constraints which can be viewed as conjunctions of binary constraints. The remaining constraints are saved in our cache: for those represented in intension, the variables instantiated in $\mathtt{s}$ are replaced by their values in the predicates, and a simplification step is performed in order to possibly reduce the representations; those represented in extension are stored explicitly; finally, for each $\mathit{weightedSum}$ constraint, the variables instantiated in $\mathtt{s}$ are replaced by their values, the constraint is simplified and only the resulting constant term needs to be stored; each $\mathit{element}$ constraint R, when binary, does not need to be stored; in the remaining case, one stores $\{\langle X_i, x_i\rangle \in \mathtt{s} \mid X_i \in S_j\}$ into the cache.

[3]This is reminiscent to the treatment of binary clauses in Dsharp, which do not need to be cached provided that unit propagation has been performed [Muise *et al.*, 2012].

Variable ordering heuristic. Our variable ordering heuristic *bc* is based on the concept of *betweenness centrality* [Brandes, 2008]which has been used in many network applications. Given a node X_i in a graph (in our case, the primal graph of the current CN in which the nodes can be identified as with the variables labelling them), $bc(X_i)$ is equal to the number of shortest paths from all nodes to all others that pass through X_i. Formally,

$$bc(X_i) = \Sigma_{X_j \neq X_i \neq X_k} \frac{\sigma_{X_i}(X_j, X_k)}{\sigma(X_j, X_k)}$$

where X_i, X_j, X_k are nodes of the given network, $\sigma(X_j, X_k)$ is the number of shortest paths from X_j to X_k, and $\sigma_{X_i}(X_j, X_k)$ are the number of those paths passing through X_i. Thus, for the CN $\mathcal{N}$ given at Example 1, X_2 is the unique variable maximizing the value of *bc*. Clearly enough, assigning first the most central variables of the primal graph $(\mathcal{X}, \mathcal{E})$ of a CN is a way to promote the generation of disjoint connected components of similar sizes, allowing the decomposition of the network into independent networks (i.e., bearing on pairwise disjoint sets of variables) of close sizes, which can be compiled separately and gathered using a $\wedge$-node in the resulting MDDG representation. Interestingly, computing the betweenness centralities of all nodes in $(\mathcal{X}, \mathcal{E})$ can be done in time $\mathcal{O}(n.p)$, where $n = \#(\mathcal{X})$ and $p = \#(\mathcal{E})$. In practice, the computation of $bc(X_i)$ for each node X_i of the primal graph $(\mathcal{X}, \mathcal{E})$ of a CN is efficient enough so that we can achieve it dynamically, i.e., for each network encountered during the compilation.

The `cn2mddg` compiler. Algorithm 1 provides the pseudo-code for the compiler cn2mddg. The compilation of a given CN $\mathcal{N}$ is achieved by calling cn2mddg on it and on the decision state $\mathtt{s} = \{\langle X_i, x_i\rangle \mid X_i \in \mathcal{X}, x_i \in D_i\}$. First of all (line 1), the arc consistency of $\mathcal{N}$ under $\mathtt{s}$ is established (the values of the variables occurring in $\mathtt{s}$ which are not supported in $\mathcal{N}$ are removed from the state). For efficiency reasons, arc consistency is ensured at start (i.e., at the first call) and then maintained dynamically each time a new elementary assignment is considered (at line 13). Then, $\mathcal{N}$ is conditioned by the resulting state (line 2).[4] At line 3, a CSP solver

[4]In the implementation, the conditioning $\mathcal{N} \mid \mathtt{s}$ at line 2 is not performed explicitly (it is presented as such for the sake of clarity); only the list of uninstantiated variables and the current domains are updated at each step (the constraints themselves are never modified for efficiency reasons).

is used to determine whether the resulting $\mathcal{N}$ is consistent or not. We developed our own solver, based on chronological backtracking and using the (now standard) conflict-directed *dom/wdeg* heuristic for selecting variables [Boussemart *et al.*, 2004]. Arc consistency is maintained at every choice step. Every constraint C_j of $\mathcal{C}$ is associated with a weight, which is incremented each time a conflict is detected. If $\mathcal{N}$ is inconsistent, then it is equivalent to the MDDG representation reduced to a leaf labelled by $\bot$, as returned by the algorithm. Line 4 concerns the other base case, when all the variables of $\mathcal{X}$ have been considered; in this situation, $\mathcal{N}$ is equivalent to the MDDG representation $\top$, as returned by the algorithm. In the remaining case (line 5), one first determines whether the current network $\mathcal{N}$ has already been encountered or not during the search. One takes advantage of the cache function which associates networks with MDDG representations given by their root nodes. If $\mathcal{N}$ has already been found, then the algorithm simply returns the root node of its MDDG compilation. Otherwise, $\mathcal{N}$ is first simplified by removing from it the universal constraints it may contain (this is achieved by the removeUniversal function, at line 6). Then (line 7) the connected components of the resulting network are looked for. The function connectedComponents returns a partition $CoCo$ of the current set of variables $\mathcal{X}$ corresponding to the connected components of the primal graph of $\mathcal{N}$ (a simple breadth-first search is performed to find them). Each element Co of $CoCo$ is considered successively (line 9); Co is a set of variables which are independent from the other elements of $CoCo$ and each network corresponding to $\mathcal{N}$ restricted to Co can be compiled separately, leading to a set of nodes $N_\wedge$ which is initialized to the empty set at line 8. For each Co, the current decision state s can be restricted to the variables occurring in Co. A variable X_i from Co is picked up using the function selectVariable at line 10. Then the values x_i from the current domain of X_i are successively considered (line 12); each of them corresponds to an elementary assignment $\{\langle X_i, x_i\rangle\}$ and the current network conditioned by s restricted to the variables of Co but X_i, and enriched with $\langle X_i, x_i\rangle$, is compiled recursively (line 13); the set $N_\vee$ of resulting nodes, initialized to the empty set at line 11 is updated at line 13. When all the values x_i have been considered, a new decision node labelled by X_i is created at line 14, and added to the set $N_\wedge$. When all the connected components of $CoCo$ have been considered, the elements of $N_\wedge$ are gathered conjunctively to form a $\wedge$-node N at line 15 thanks to the function aNode. This node is added to the cache associated with the entry $\mathcal{N}$ (line 16) and finally returned (line 17) as the root node of the MDDG representation of $\mathcal{N}$.

Algorithm 1 is guaranteed to terminate since at each recursive step at least one variable of the initial CN is instantiated. By construction, the resulting MDDG representation is equivalent to $\mathcal{N} \mid$ s.

Algorithm 1: cn2mddg

```
input  : a constraint network $\mathcal{N} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$
input  : a decision state s over a subset of $\mathcal{X}$
output: the root node $N$ of an MDDG representation
1  s ← ac($\mathcal{N}$, s)
2  $\mathcal{N} \leftarrow \mathcal{N} \mid$ s
3  if unsat($\mathcal{N}$) then return leaf($\bot$)
4  if $\#(\mathcal{X}) = 0$ then return leaf($\top$)
5  if cache($\mathcal{N}$) $\neq$ nil then return cache($\mathcal{N}$)
6  $\mathcal{C}$ ← removeUniversal($\mathcal{C}$)
7  $CoCo$ ← connectedComponents($\mathcal{N}$)
8  $N_\wedge \leftarrow \emptyset$
9  foreach $Co \in CoCo$ do
10     $X_i$ ← selectVariable($Co$)
11     $N_\vee \leftarrow \emptyset$
12     foreach $x_i$ s.t. $\langle X_i, x_i\rangle \in$ s do
13         $N_\vee \leftarrow N_\vee \cup$ cn2mddg($\mathcal{N}$, s[$Co \setminus \{X_i\}$] $\cup \{\langle X_i, x_i\rangle\}$)
14     $N_\wedge \leftarrow N_\wedge \cup$ dNode($X_i, N_\vee$)
15 $N$ ← aNode($N_\wedge$)
16 cache($\mathcal{N}$) ← $N$
17 return $N$
```

5 Experiments

While compiling CNs has been an issue considered for years (see e.g., [Vempaty, 1992; Amilhastre *et al.*, 2002]), we are not aware of any available compiler suited to CNs over non-Boolean domains and handling constraints which are intensionally represented. Especially, the AOMDD compiler available at http://graphmod.ics.uci.edu/group/aomdd assumes that each constraint of the input CN is represented extensionally by the list of its satisfying tuples. Fortunately, many SAT-encodings of CNs have been pointed out so far (see among others [de Kleer, 1989; Iwama and Miyazaki, 1994; Walsh, 2000; Gent, 2002]), rendering feasible a comparison with Decision-DNNF compilations of CNF translations of such CNs.

Setup. We have considered 173 CNs from 15 data sets, downloaded from github.com/MiniZinc/minizinc-benchmarks, www.cril.univ-artois.fr/~lecoutre/benchmarks.html, and www.itu.dk/research/cla/externals/clib/. Those data sets correspond to several families of problems, including configuration problems, scheduling problems, frequency allocation problems. For some instances, the constraints are represented extensionally, by the list of satisfying tuples or by the list of forbidden tuples; for other instances, they are given in intension.

Our purpose was to compile each input CN into an MDDG representation using cn2mddg, and into Decision-DNNF representations, using first a translation of it into CNF, then the Decision-DNNF compiler Dsharp. Two CNF encodings have been considered in our experiments: the sparse encoding S of the domains together with a mixed clause encoding of the constraints (i.e., each constraint is encoded using the support encoding or the conflict encoding, in order to minimize the number of generated clauses), and the log encoding L of the domains together with a conflict encoding of the constraints.[5]

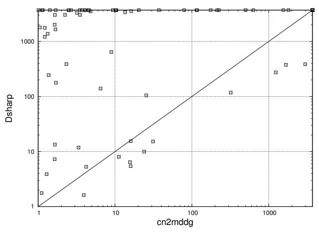
[5]Some of the available translators, like Sugar [Tamura *et al.*, 2009] or Azucar [Tanjo *et al.*, 2012] lead to encodings which do not preserve equivalence (they are oriented to solve the satisfaction

	CN								CNF - sparse mixed encoding				CNF - log conflict encoding			
Name	type	$\#\mathcal{X}$	$\#C$	maxA	maxD	tw	time	size	#pv	#pcl	time	size	#pv	#pcl	time	size
rect-packing/rect-packingrpp09-true	I	2196	2353	10	36	19	1673.33	514754	37044	593518	375.66	16118647	4466	392657	TO	-
ghoulomb/ghoulomb3-4-5	I	2033	2051	11	26	31	15.17	5162	MO	MO	MO	MO	MO	MO	MO	MO
driver/normalized-driverlogw-08c-sat-ext	E	408	9321	2	11	92	15.63	2931	9528	62825	6.42	139306	1050	46081	32.78	499796
scheduling/talent-concert	I	325	352	46	316	52	1277.21	404437	MO	MO	MO	MO	MO	MO	MO	MO
fapp/fapp19/normalized-fapp19-0350-6	I	350	3114	2	802	130	79.34	1694146	166130802	867243022	-	MO	MO	MO	MO	MO
costaArray/CostasArray10	I	110	338	4	19	23	10.39	13440	149564	841930	TO	-	540	3606946	TO	-
costaArray/CostasArray14	I	210	808	4	27	36	TO	-	988671	5568047	TO	-	1036	34687218	-	MO
photo/photophoto2	I	89	133	21	11	21	499.93	9564220	685555	14326576	TO	-	204	10923133	-	MO
rlfap/normalized-scen4	I	680	3967	2	44	30	3.47	52226	915553	4875002	-	MO	4060	3058032	TO	-
radiation/radiation04	I	781	569	9	5180	33	-	MO	MO	MO	MO	MO	MO	MO	MO	MO
renault/normalized-renault-mod-32-ext	E	111	154	10	42	11	20.39	160238	222582	1755876	TO	-	286	138124077	-	MO
renault/normalized-renault-mod-11-ext	E	111	149	10	42	10	16.22	41919	223718	1762294	3538.01	2399273	286	138117804	-	MO
still-life/still-life7x7	I	690	803	50	50	49	1819.87	738478	MO	MO	MO	MO	MO	MO	MO	MO
configit/Aralia/edfpa15r	I	198	110	13	2	28	175.49	2044261	396	24710	-	MO	396	24710	-	MO
configit/Aralia/edfpa14q	I	505	194	22	2	34	TO	-	1010	5793030	TO	-	1010	5793030	TO	-
configit/Aralia/das9207	I	600	324	8	2	15	8.94	45853	1200	3894	642.68	22707412	1200	3894	97.86	9937506

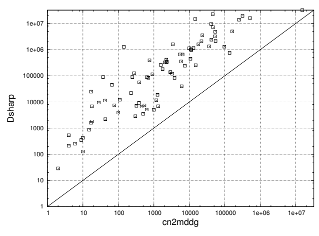
Table 1: An excerpt of our empirical results.

For each instance, we computed the compilation time (in seconds) and the size of the compiled representation (number of arcs in the DAG). For translation-based approaches, we also computed the translation time and the size of the resulting `CNF` formula (number of variables and number of clauses). Our experiments have been conducted on a Quadcore Intel XEON X5550 with 32GB of memory. A time limit of 3600s for the `CNF` translation phase (resp. the off-line compilation phase) and a total amount of 8GB of memory for storing the resulting `CNF` formula (resp. the compiled representation) have been considered for each instance.

Some Results. Within the time and memory limits we set, `cn2mddg` succeeded in compiling 131 instances over 173; the computation aborted with a time-out (TO) for 32 instances, and with a memory-out (MO) for 10 instances. This heavily contrasts with `Dsharp` which succeeded in compiling only 83 instances when the S encoding was used, and 61 instances when the L encoding was used. More in details, the `CNF` translation using S (resp. L) led to a memory-out for 27 (resp. 35) instances over 173; over the 146 (resp. 138) remaining instances, the `DNNF` compilation using `Dsharp` aborted with a time-out for 24 (resp. 21) instances, and with a memory-out for 39 (resp. 56) instances.



(a) Time



(b) Size

Figure 3: `cn2mddg` vs. `Dsharp`: comparing the compilation times and the sizes of the compiled forms.

Whatever the encoding used (S or L), the translate-then-compile approach appears as impractical in many cases. This can be explained both by the huge number of Boolean variables in the generated `CNF` formulae, and the structure loss inherent to the `CNF` format (compared to the CN one). Contrastingly, our compiler `cn2mddg` proved much more robust since it succeeded in compiling many CNs which are out of reach of the translate-then-compile approaches. Especially, each of the 83 instances which have been compiled with success using `Dsharp` (with the S encoding) have also proved compilable into `MDDG` using `cn2mddg`. The compilation times required to produce `MDDG` representations from the input network are often smaller than the compilation times required to produce `DNNF` representations from the `CNF` translation of the network. More importantly, the sizes of the resulting compiled representations turn out to be always smaller, sometimes by several orders of magnitude, when `MDDG` is targeted compared to the `DNNF` case. This is salient on Figure 3, where each dot represents one of the 83 instances for which `Dsharp` (with the S encoding) did not fail. The time needed to compute (resp. the size of) the resulting `MDDG` representation is given by its x-coordinate and the the time needed to compute (resp. the size of) the resulting `DNNF` representation from the `CNF` translation is given by its y-coordinate.[6] Every scale is a logarithmic one.

Table 1 presents a selection of the results. Each line corresponds to a constraint network $\mathcal{N} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$ identified by the leftmost column. The next columns give respectively the "type" of CN (Extension or Intension), the number $\#\mathcal{X}$ of variables, the number $\#C$ of constraints, the maximal arity $maxA$ of the constraints, the maximal size $maxD$ of the domains, a upper bound tw of the treewidth of its primal graph,[7] the time needed to get the `MDDG` representation using `cn2mddg`, and the size of it. For the two `CNF` encodings under consideration, one can find the number $\#pv$ of propositional variables in it, the number $\#pcl$ of clauses in it, the time needed to get the `DNNF` representation using `Dsharp`, and the size of it. The reported results illustrate the benefits offered by `cn2mddg` over the translate-then-compile ap-

problem), and cannot be used as such for our compilation purpose.

[6]Note that the time differences in favor of `cn2mddg` would be even larger if the time needed to translate the CN into `CNF` would have been taken into account.

[7]Computed using `QuickBB` – see `http://www.hlt.utdallas.edu/~vgogate/quickbb.html` – equipped with the random ordering heuristic and for an allocated time of 1800s.

proaches, about both the number of benchmarks for which the compilation succeeded, and the sizes of the compiled representations. They also show the feasibility of `MDDG` compilation of CNs corresponding to real applications, and of significant complexity (often out of reach of tree clustering compilations [Dechter and Pearl, 1989], given the sizes of their domains and the treewidth of their primal graphs).

We also performed a differential evaluation for assessing the impact of each technique used within `cn2mddg`. Let `dom/wdeg+noU` be the version of `cn2mddg` for which the *dom/wdeg* heuristic is used (instead of *bc*), and the handling of universal constraints is disabled. `dom/wdeg+noU` solved only 101 instances (over 173) within the time and memory limits. Besides, the number of instances (over 101) for which the size of the `MDDG` representation obtained by `cn2mddg` (resp. `dom/wdeg+noU`) is lower than $p = \frac{1}{2}$ times the size of the `MDDG` representation obtained by `dom/wdeg+noU` (resp. `cn2mddg`) is 35 (resp. 6). The corresponding number for the proportion $p = \frac{1}{10}$ (instead of $\frac{1}{2}$) is 12 (resp. 0).

6 Conclusion

The contribution of the paper is a top-down algorithm `cn2mddg` for compiling finite-domain CNs into multivalued decomposable decision graphs. `cn2mddg` takes advantage of a specific caching technique, a new variable ordering heuristic based on betweenness centrality, and the handling of universal constraints. Intensive experiments showed that `cn2mddg` succeeds in compiling CNs which cannot be compiled into Decision-`DNNF` via a preliminary translation into `CNF`, and leads to compiled forms which are typically much smaller.

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